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THE

THEORY OF EQUATIONS

WITH AN

INTRODUCTION TO THE THEORY OF BINARY ALGEBRAIC FORMS

BY THE LATE

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NOTE.

When the editing of this volume was undertaken, pages 1–48 and 65–128 had been stereotyped, and so no alteration was made in them except to insert the usual symbols for partial differentiation. Some notes are given on these pages at the end of the volume on pages 306, 307. The following Articles or

Note.

Examples were re-written or much revised or considerably increased: Art. 186; Ex. 8, p. 166; Ex. 9, p. 168; Art. 201; Art. 202; Art. 205; Art. 206; Art. 207; Art. 211; Ex. 15, p. 208; Ex. 17, p. 208; Ex. 18, p. 209; Ex. 19, p. 210; Ex. 20, p. 210; Art. 216 (on Unique Ternary form such that concomitants are the same for Binary and derived Ternary quantics); Ex. 7, p. 237; Ex. 8, p. 238; Ex. 10, p. 239; Ex. 13, p. 241; Art. 226; Art. 227; Art. 234 (on the group of an equation, pp. 276, 277); Ex. 1, p. 278; Ex. 2, p. 279.

Minor alterations or additions were made on pp. 134, 143, 216, 217, 224, 226, 254, 262, 267, 268, 269, 270.

A section on Abelian equations has been added, pp. 296–305.

M. W. J. FRY,
Editor.

CHAPTER XIII.

DETERMINANTS.

127. **Elementary Notions and Definitions.**—This chapter will be occupied with a discussion of an important class of functions which constantly present themselves in analysis. These functions possess remarkable properties, by a knowledge of which much simplification can be introduced into many operations in both pure and applied mathematics.

The function $a_1b_2 + a_2b_1$, of the four quantities

$$\begin{matrix} a_1, & b_1, \\ a_2, & b_2, \end{matrix}$$

is obtained by assigning to a and b, written in alphabetical order, the suffixes 1, 2, and 2, 1, corresponding to the two permutations of the numbers 1, 2, and adding the two products so formed.

Similarly, the function

$$a_1b_2c_3 + a_1b_3c_2 + a_2b_3c_1 + a_2b_1c_3 + a_3b_1c_2 + a_3b_2c_1, \qquad (1)$$

of the nine quantities

$$\begin{matrix} a_1, & b_1, & c_1, \\ a_2, & b_2, & c_2, \\ a_3, & b_3, & c_3, \end{matrix}$$

is obtained by adding all the products abc which can be formed by assigning to the letters (retained in their alphabetical order) suffixes corresponding to all the permutations of the numbers 1, 2, 3. The whole expression might be represented by (abc), or any other convenient notation, from which all the terms could be written down.

The notation $(abcd)$ might be employed to represent a similar function of the 16 quantities a_1, b_1, c_1, d_1, a_2, &c., consisting of 24 terms, which can all be written down by the aid of the 24 permutations of the numbers 1, 2, 3, 4.

And, in general, taking n letters $a, b, c, \ldots l$, we can write down a similar function consisting of $n(n-1)(n-2) \ldots 3.2.1$ terms, this being the number of permutations of the first n numbers, $1, 2, 3 \ldots n$.

Now the functions above referred to, which are of such frequent occurrence in mathematical analysis, differ from those just described in one respect only, viz.: of the $1.2.3 \ldots n$ (which is an even number) terms, half are affected with positive, and half with negative signs, instead of being all positive, as in the expression written down on the preceding page.

We shall now give some instances of the functions which will be discussed in this chapter. They occur most frequently as the result of elimination from linear equations. If, for example, x and y be eliminated from the equations

$$a_1x + b_1y = 0, \quad a_2x + b_2y = 0,$$

the result is $$a_1b_2 - a_2b_1 = 0.$$

Again, the result of eliminating x, y, z from the equations

$$a_1x + b_1y + c_1z = 0,$$
$$a_2x + b_2y + c_2z = 0,$$
$$a_3x + b_3y + c_3z = 0,$$

is, as the student will readily perceive by solving from two of the equations and substituting in the third,

$$a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 = 0; \qquad (2)$$

and this function differs from (1) given on the preceding page only in having three of its terms negative, instead of having the six terms positive.

Similarly, the process of elimination from four linear equations gives rise to a function of the sixteen quantities a_1, b_1, c_1,

d_1, $a_2\ b_3$, &c., which differs from the function above represented by $(abcd)$ only in having twelve of its terms negative.

Expressions of the kind here described are called *Determinants.** The notation by which they are usually represented was first employed by Cauchy, and possesses many advantages in the treatment of these expressions. The quantities of which the function consists are arranged in a square between two vertical lines. For example, the notation

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

represents the determinant $a_1b_2 - a_2b_1$.

Similarly, the expression on the left-hand side of equation (2) is represented by the notation

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

And, in general, the determinant of the n^2 quantities $a_1, b_1, c_1 \ldots l_1, a_2, b_2$, &c., is represented by

$$\begin{vmatrix} a_1 & b_1 & c_1 & . & . & . & l_1 \\ a_2 & b_2 & c_2 & . & . & . & l_2 \\ a_3 & b_3 & c_3 & . & . & . & l_3 \\ . & . & . & . & . & . & . \\ a_n & b_n & c_n & . & . & . & l_n \end{vmatrix}. \qquad (3)$$

By taking the n letters in alphabetical order, and assigning to them suffixes corresponding to the $n(n-1)(n-2)\ldots 3.2.1$ permutations of the numbers $1, 2, 3, \ldots n$, all the terms of the determinant can be written down. Half of the terms must receive positive, and half negative signs. In the next Article

* See Note A at the end of Vol. II.

the rule will be given by which the positive and negative terms are distinguished.

The individual letters a_1, b_1, c_1, ... a_2, ... &c., of which a determinant is composed, are called *constituents*, and by some writers *elements*. Any series of constituents such as a_1, b_1, c_1, ... l_1, arranged horizontally, form a *row* of the determinant; and any series such as a_1, a_2, a_3, ... a_n, arranged vertically, form a *column*. The term *line* will sometimes be used to express a row or column indifferently.

128. **Rule with regard to Signs.**—It is evident from the preceding Article that each term of the determinant will, since it contains all the letters, contain one constituent (and only one) from every column; and will also, since the suffixes in each term comprise all the numbers, contain one constituent (and only one) from every row. We may therefore regard the square array (3) of Art. 127 as the symbolical representation of a function consisting in general of $n(n-1)(n-2) \ldots 3.2.1$ terms, comprising all possible products which can be formed by taking one constituent and one only from each row, and one constituent and one only from each column. All that is required to give perfect definiteness to the function is to fix the sign to be attached to any particular term. For this purpose the following two rules are to be observed:—

(1). *The term $a_1b_2c_3 \ldots l_n$, formed by the constituents situated in the diagonal drawn from the left-hand top corner to the right-hand bottom corner, is positive.*

This is called the *leading* or *principal* term. In it the suffixes and letters both occur in their natural order; and from it the sign of any other term is obtained by the following rule:—

(2). *The sign of any other term is positive or negative, according as it contains among its suffixes an even or odd number of inversions of order as compared with the suffixes of the leading term.*

The letters are supposed to retain the alphabetical order, and an "inversion" is said to occur whenever any higher number precedes a lower among the suffixes. In the term $a_3b_4c_1d_2$, for example, there are four inversions, the number 3 occurring before 1 and before 2, and 4 occurring before 1 and before 2. Similarly, $a_3b_2c_5d_4e_1$ contains six inversions, as the student will readily perceive. The following will be found to be a useful modification of this rule:—*A transposition (or interchange) of two adjacent suffixes alters the sign of a term.* For it is easy to see that any such transposition is equivalent to the gain or loss of one inversion. No inversion, in fact, in the series is disturbed by the process, except such as depends on the relative position of the two adjacent suffixes when compared with one another. If before transposition these suffixes are in their natural order, one inversion is gained by the process; but if not, one is lost. In the arrangement 5342716, for example, the interchange of 2 and 7 introduces one additional inversion, the number being thus increased from eleven to twelve. The sign of the corresponding term in the determinant is therefore altered from − to +.

It is easy now to justify the remark in Art. 127, that a determinant contains an equal number of positive and negative terms. For, from any term another can be derived which differs from the first only by the transposition of the last two suffixes, and these are the only two terms which agree in the order of permutation of the first $n-2$ suffixes. All the terms, therefore, can be arranged in pairs such that if the first is positive, the second is negative, and *vice versa*.

EXAMPLES.

1. What is the sign of the term $a_3b_4c_2d_5e_1$ in the determinant of the 5th order?

The question is, How many inversions of order occur in 34251; or, How many interchanges will change the order 12345 into 34251? Here, when 3 is interchanged with 2, and afterwards with 1, it comes into the leading place, the order becoming 31245. Again, the interchange in 31245 of 4 with 2, and afterwards with 1, presents the order 34125. The interchange of 2 with 1 gives the order 34215; and finally, the interchange of 5 with 1 gives the required order 34251. Thus there are in all six interchanges; and therefore the required sign is positive.

The general mode of proceeding may plainly be stated as follows:—Take the figure which stands first in the required order, and move it from its place in the natural order 1234 . . . into the leading place, counting one displacement for each figure passed over. Take then the figure which stands second in the required order, and move it from its place in the natural order into the second place; and so on. If the number of displacements in this process be even, the sign is positive; if it be odd, the sign is negative.

2. What is the sign of the term $a_3b_7c_6d_5e_1f_4g_2$ in the determinant of the 7th order?

Here two displacements bring 3 to the leading place; five displacements then bring 7 to the second place; four then bring 6 to the third place; three then bring 5 to the fourth place; the figure 1 is in its place; and finally, one displacement brings 4 into the sixth place. Thus there are in all fifteen displacements; and the required sign is therefore negative.

3. Write down all the terms of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

The six permutations of suffixes in which the figure 1 occurs first are

1234, 1243, 1324, 1342, 1423, 1432.

The six corresponding terms are, as the student will easily see by applying the Rule (2), as in the previous examples,

$$a_1b_2c_3d_4 - a_1b_2c_4d_3 + a_1b_3c_4d_2 - a_1b_3c_2d_4 + a_1b_4c_2d_3 - a_1b_4c_3d_2.$$

The other eighteen terms, corresponding to the permutations in which the figures 2, 3, 4, respectively, stand first, are as follows:—

$$a_2b_1c_4d_3 - a_2b_1c_3d_4 + a_2b_3c_1d_4 - a_2b_3c_4d_1 + a_2b_4c_3d_1 - a_2b_4c_1d_3$$
$$+ a_3b_1c_2d_4 - a_3b_1c_4d_2 + a_3b_2c_4d_1 - a_3b_2c_1d_4 + a_3b_4c_1d_2 - a_3b_4c_2d_1$$
$$+ a_4b_1c_3d_2 - a_4b_1c_2d_3 + a_4b_2c_1d_3 - a_4b_2c_3d_1 + a_4b_3c_2d_1 - a_4b_3c_1d_2.$$

4. Show that any interchange of two suffixes (the letters retaining their order) alters the sign of a term.

For, if there are m elements between the two whose suffixes are interchanged, the proposed transposition can be effected by $2m + 1$ transpositions of adjacent suffixes. By the aid of this proposition the sign of a term can usually be found by a smaller number of transpositions than is required in the general method described in Ex. 1. Thus, in Ex. 2, five transpositions are enough to fix the sign of the term, viz.: first, of 1 with 3, and then in succession 1 with 6, 1 with 4, 1 with 5, and finally 2 with 7. The determination of the smallest number of transpositions necessary for this purpose is easily shown to depend on an elementary proposition in the theory of substitutions. (Compare Chap. XX. of this Vol.)

5. Show that any interchange of two letters, the order of the suffixes being retained, alters the sign of a term.

For, if two letters be interchanged, and the corresponding elements then interchanged, the entire process is equivalent to an interchange of suffixes. If, for example, in $a_1b_2c_3d_4e_5$, b and e be interchanged, we derive $a_1e_2c_3d_4b_5$, which is equal to $a_1b_5c_3d_4e_2$; and this is derived from the given term by transposition of the suffixes 2 and 5.

6. Show that if any two adjacent figures be moved together over any number m of figures, the sign is unaltered.

For if they be moved separately, the whole process is equivalent to a movement over $2m$ figures.

7. Determine the sign to be attached to the second diagonal term, viz. $a_nb_{n-1}c_{n-2}\ldots k_2l_1$, in the determinant of the n^{th} order.

Here the number of inversions of order is clearly

$$(n-1)+(n-2)+(n-3)+\ldots+2+1=\frac{n(n-1)}{2}.$$

Hence the required sign is $(-1)^{\frac{n(n-1)}{2}}$.

129. In the propositions of the present and following Articles are contained the most important elementary properties of determinants which, by the aid of Cauchy's notation above described, render the employment of these functions of such practical advantage.

Prop. I.—*If any two rows, or any two columns, of a determinant be interchanged, the sign of the determinant is changed.*

This follows at once from the mode of formation (Rule (2), Art. 128); for an interchange of two rows is the same as an interchange of two suffixes, and an interchange of two columns is the same as an interchange of two letters; so that in either case the sign of every term of the determinant is changed. (See Exs. 4 and 5, Art. 128.)

By aid of this proposition the rule for obtaining the sign of any term may be stated in a form which is usually more convenient for practical purposes than that already given. It will readily be perceived that the general mode of procedure explained in Ex. 1, Art. 128, is equivalent to the following.

Bring by movements of rows (or columns) the constituents of the term whose sign is required into the position of the leading diagonal. The sign of the term will be positive or negative according as the number of displacements is even or odd.

EXAMPLE.

What sign is to be attached to the term $\lambda\beta nx$ in the determinant

$$\begin{vmatrix} a & b & c & \underline{x} \\ \alpha & \underline{\beta} & \gamma & y \\ l & m & \underline{n} & c \\ \underline{\lambda} & \mu & \nu & 0 \end{vmatrix} ?$$

Here a movement of the fourth row over three rows (*i. e.* three displacements) brings λ into the leading place. One displacement of the original second row upwards brings β into the required place in the diagonal term. And one further displacement of the original third row upwards effects the required arrangement, bringing $\lambda\beta nx$ into the diagonal place. Thus the number of displacements being odd, the required sign is negative.

130. PROP. II.—*Whenever, in any determinant, two rows or two columns are identical, the determinant vanishes.*

For, by Prop. I., the interchange of these two lines ought to change the sign of the determinant Δ; but the interchange of two identical rows or columns cannot alter the determinant in any way; hence $\Delta = -\Delta$, or $\Delta = 0$.

131. PROP. III.—*The value of a determinant is not altered if the rows be written as columns, and the columns as rows.*

For all the terms, formed by taking one constituent from each row and one from each column, are plainly the same in value in both cases; the principal term is identically the same; and to determine the sign of any other term (by Prop. I.) the number of displacements of rows necessary to bring it into the leading diagonal in the first case is the same as the number of displacements of columns necessary in the second case.

Example.

$$\begin{vmatrix} a_1 & b_1 & \underline{c_1} & d_1 \\ \underline{a_2} & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & \underline{d_3} \\ a_4 & \underline{b_4} & c_4 & d_4 \end{vmatrix} = \begin{vmatrix} a_1 & \underline{a_2} & a_3 & a_4 \\ b_1 & b_2 & b_3 & \underline{b_4} \\ \underline{c_1} & c_2 & c_2 & c_4 \\ d_1 & d_2 & \underline{d_3} & d_4 \end{vmatrix}.$$

Here the sign of any term, e.g. $a_2b_4c_1d_3$, is the same in both determinants. For three displacements of rows are required to bring this term into the leading position in the first determinant; and the same number of displacements of columns is required to bring the same constituents into the leading position in the second determinant.

132. Prop. IV.—*If every constituent in any line be multiplied by the same factor, the determinant is multiplied by that factor.*

For every term of the determinant must contain one, and only one, constituent from any row or any column.

Cor. 1. If the constituents in any line differ only by the same factor from the constituents in any parallel line, the determinant vanishes.

Cor. 2. If the signs of all the constituents in any line be changed, the sign of the determinant is changed. For this is equivalent to multiplying by the factor -1.

Examples.

1. $$\begin{vmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

2. $$\begin{vmatrix} \alpha_1 & m\alpha_1 & \alpha_2 \\ \beta_1 & m\beta_1 & \beta_2 \\ \gamma_1 & m\gamma_1 & \gamma_2 \end{vmatrix} = m \begin{vmatrix} \alpha_1 & \alpha_1 & \alpha_2 \\ \beta_1 & \beta_1 & \beta_2 \\ \gamma_1 & \gamma_1 & \gamma_2 \end{vmatrix} = 0.$$

3. Show that the following determinant vanishes:—

$$\begin{vmatrix} 3 & 1 & 5 & 2 \\ 2 & 5 & 7 & 3 \\ 8 & 9 & 1 & 4 \\ 6 & 15 & 21 & 9 \end{vmatrix}.$$

4. Prove the identity

$$\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}.$$

Represent the first determinant by Δ, and multiply the rows by a, b, c, respectively. We have then

$$abc\Delta = \begin{vmatrix} abc & a^2 & a^3 \\ abc & b^2 & b^3 \\ abc & c^2 & c^3 \end{vmatrix};$$

and, dividing the first column by abc, the result follows.

5. Prove the identity

$$\begin{vmatrix} \beta\gamma\delta & \alpha & \alpha^2 & \alpha^3 \\ \gamma\delta\alpha & \beta & \beta^2 & \beta^3 \\ \delta\alpha\beta & \gamma & \gamma^2 & \gamma^3 \\ \alpha\beta\gamma & \delta & \delta^2 & \delta^3 \end{vmatrix} = \begin{vmatrix} 1 & \alpha^2 & \alpha^3 & \alpha^4 \\ 1 & \beta^2 & \beta^3 & \beta^4 \\ 1 & \gamma^2 & \gamma^3 & \gamma^4 \\ 1 & \delta^2 & \delta^3 & \delta^4 \end{vmatrix}.$$

6. Prove

$$\begin{vmatrix} 2 & 1 & -7 \\ -4 & -3 & 8 \\ 6 & 5 & -9 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 7 \\ 2 & 3 & 8 \\ 3 & 5 & 9 \end{vmatrix}.$$

Change all the signs of the second row, and afterwards of the third column.

7. Prove

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} = \frac{1}{\alpha\beta\gamma}\begin{vmatrix} 1 & 1 & 1 \\ \alpha'\beta\gamma & \beta'\gamma\alpha & \gamma'\alpha\beta \\ \alpha''\beta\gamma & \beta''\gamma\alpha & \gamma''\alpha\beta \end{vmatrix}.$$

This is easily proved by multiplying the columns of the first determinant by $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$, respectively; and then dividing the first row by $\alpha\beta\gamma$.

It is evident that a similar process may be employed to reduce any determinant to one in which all the constituents of any selected row or column shall be units.

8. Reduce the following determinant to one in which the first row shall consist of units :—

$$\Delta = \begin{vmatrix} 4 & 2 & 5 & 10 \\ 1 & 1 & 6 & 3 \\ 7 & 3 & 0 & 5 \\ 0 & 2 & 5 & 8 \end{vmatrix}.$$

Since 20 is the least common multiple of 4, 2, 5, 10, it is sufficient to multiply the columns in order by 5, 10, 4, 2; we thus obtain

$$\Delta = \frac{1}{5 \,.\, 10 \,.\, 4 \,.\, 2} \begin{vmatrix} 20 & 20 & 20 & 20 \\ 5 & 10 & 24 & 6 \\ 35 & 30 & 0 & 10 \\ 0 & 20 & 20 & 16 \end{vmatrix}.$$

Taking out the multiplier 20 from the first row, 5 from the third row, and 4 from the fourth row, we get finally

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 5 & 10 & 24 & 6 \\ 7 & 6 & 0 & 2 \\ 0 & 5 & 5 & 4 \end{vmatrix}.$$

9. Prove the identity

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix} \equiv (\beta - \gamma)(\gamma - \alpha)(\alpha - \beta).$$

Since if β were equal to γ, two columns would become identical, $\beta - \gamma$ must be a factor in the determinant. Similarly, $\gamma - \alpha$ and $\alpha - \beta$ must be factors in it. Hence the product of the three differences can differ by a numerical factor only from the value of the determinant, since both functions are of the third degree in α, β, γ; and by comparing the term $\beta\gamma^2$ we observe that this factor is $+ 1$.

10. Prove similarly the identity

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix} \equiv -(\beta - \gamma)(\alpha - \delta)(\gamma - \alpha)(\beta - \delta)(\alpha - \beta)(\gamma - \delta).$$

It is evident that a similar proof shows in general that the value of the determinant of this form, constituted by the n quantities $\alpha, \beta, \gamma \ldots \lambda$, is the product of the $\frac{1}{2}n(n-1)$ differences which can be formed with these n quantities.

133. **Minor Determinants. Definitions.**—When in a determinant any number of rows and the same number of columns are suppressed, the determinant formed by the remaining constituents (maintaining their relative positions) is called a *minor determinant.*

If one row and one column only be suppressed, the corresponding minor is called a *first minor*. If two rows and two columns be suppressed, the minor is called a *second minor*; and so on. The suppressed rows and columns have common constituents forming a determinant; and the minor which remains is said to be *complementary* to this determinant. The minor complementary to the leading constituent a_1 is called the *leading first minor*, and its leading first minor again is the *leading second minor* of the original determinant.

It is usual to denote a determinant in general by Δ. We shall denote by Δ_a the first minor obtained by suppressing in Δ the row and column which contain any constituent a; by $\Delta_{a,\beta}$ the second minor obtained by suppressing the two rows and two columns which contain a and β; and so on. Thus Δ_{a_1} represents the leading first minor, and Δ_{a_1, b_2} or Δ_{a_2, b_1} the leading second minor.

The determinant Δ, formed by the constituents a_1, b_1, c_1, &c., is often denoted for brevity by placing the leading term within brackets as follows:—

$$\Delta = (a_1 b_2 c_3 \ldots\ldots l_n).$$

The notation $\Sigma \pm a_1 b_2 c_3 \ldots l_n$ is also used to represent Δ; this expressing its constitution as consisting of the sum of a number of terms (with their proper signs attached) formed by taking all possible permutations of the n suffixes.

134. **Development of Determinants.**—Since every term of any determinant contains one, and only one, constituent from each row and from each column, it follows that Δ *is a linear and homogeneous function of the constituents of any one row or any one*

column. We may therefore write

$$\Delta = a_1A_1 + a_2A_2 + a_3A_3 + \&c.,$$
$$\Delta = b_1B_1 + b_2B_2 + b_3B_3 + \&c.;$$

or, again,

$$\Delta = a_1A_1 + b_1B_1 + c_1C_1 + \&c.,$$
$$\Delta = a_2A_2 + b_2B_2 + c_2C_2 + \&c.$$

The student, on referring to Ex. 3, Art. 128, will observe that the determinant of the fourth order there written at length is constituted in the way here described, namely,

$$\Delta = a_1\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} + a_2\begin{vmatrix} b_1 & c_1 & d_1 \\ b_4 & c_4 & d_4 \\ b_3 & c_3 & d_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} + a_4\begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_2 & c_2 & d_2 \end{vmatrix}$$

We proceed to show that in the general case, writing Δ in the form

$$\Delta = a_1A_1 + a_2A_2 + a_3A_3 + \ldots + a_nA_n,$$

the coefficients A_1, A_2, A_3, &c., are determinants of the order $n-1$.

In effecting all the permutations of the suffixes $1, 2, 3 \ldots . n$, suppose first 1 to remain in the leading place, as in the example referred to; we then obtain $1.2.3 \ldots . (n-1)$ terms which have a_1 as a factor, and

$$a_1A_1 = a_1\Sigma \pm b_2c_3 \ldots l_n;$$

hence

$$A_1 = \Sigma \pm b_2c_3 \ldots l_n = \begin{vmatrix} b_2 & c_2 & \ldots & l_2 \\ b_3 & c_3 & \ldots & l_3 \\ . & . & . & . \\ b_n & c_n & \ldots & l_n \end{vmatrix};$$

and this determinant is the minor corresponding to the constituent a_1, or $A_1 = \Delta_a$.

To find the value of A_2, we bring a_2 into the leading place by one displacement of rows. This changes the sign of Δ, so that we obtain $A_2 = -\Delta_{a_2}$, i.e. A_2 = the minor corresponding to a_2 with its sign changed. Again, bringing a_3 to the leading place by two displacements, we have $A_3 - \Delta_{a_3}$; and so on.

Thus we conclude that, in general,

$$\Delta = a_1\Delta_{a_1} - a_2\Delta_{a_2} + a_3\Delta_{a_3} - a_4\Delta_{a_4} + \&c.$$

Similarly, we can expand Δ in terms of the constituents of any other column, or any row. For example,

$$\Delta = a_1\Delta_{a_1} - b_1\Delta_{b_1} + c_1\Delta_{c_1} - \&c.$$

If it be required to obtain the proper sign to be attached to the minor which multiplies any constituent in the expanded form, we have only to consider how many displacements would bring that constituent to the leading place. For example, suppose the determinant $(a_1b_2c_3d_4e_5)$ is expanded in terms of its fourth column, and that it is required to find what sign is to be attached to $d_2\Delta_{d_2}$. Here two displacements upwards, and afterwards three to the left, will bring d_2 to the leading place; hence the sign is negative. This rule may be stated simply as follows: *Proceed from a_1 to the constituent under consideration along the top row, and down the column containing the constituent; the number of letters passed over before reaching the constituent will decide the sign to be attached to the minor.* In the example just given, beginning at a_1, we count a_1, b_1, c_1, d_1, d_2, i. e. five; and this number being odd, the required sign is negative.

It will be found convenient to retain both notations here employed for the development of a determinant. The expansion in terms of the minors, with signs alternately positive and negative, is useful in calculating the value of a determinant by successive reductions to determinants of lower degree. For some purposes, as will appear in the Articles which follow, it is more convenient to employ the notation first given, in which the signs are all positive (whatever the row or column under consideration) and the coefficient (or co-factor) of any constituent represented by the corresponding capital letter. By substituting for the capital letter the corresponding minor with the proper sign, determined in the manner above explained, the latter notation is changed into the former.

Examples.

1. $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1.$$

(Compare (2), Art. 127.)

2. $$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = a \begin{vmatrix} b & f \\ f & c \end{vmatrix} - h \begin{vmatrix} h & g \\ f & c \end{vmatrix} + g \begin{vmatrix} h & g \\ b & f \end{vmatrix}$$

$$= abc + 2fgh - af^2 - bg^2 - ch^2.$$

3. Expand the determinant of the fourth order in terms of the constituents of the fourth row.

$$\Delta = -a_4\Delta_{a_4} + b_4\Delta_{b_4} - c_4\Delta_{c_4} + d_4\Delta_{d_4}.$$

$$= -a_4 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} + b_4 \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} - c_4 \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} + d_4 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

When the determinants of the third order are expanded, this will give the expression of Ex. 3, Art. 128, as the student will easily verify.

4. $$\begin{vmatrix} 3 & 2 & 4 \\ 7 & 6 & 1 \\ 5 & 3 & 8 \end{vmatrix} = 3 \begin{vmatrix} 6 & 1 \\ 3 & 8 \end{vmatrix} - 7 \begin{vmatrix} 2 & 4 \\ 3 & 8 \end{vmatrix} + 5 \begin{vmatrix} 2 & 4 \\ 6 & 1 \end{vmatrix}$$

$$= 3(48 - 3) - 7(16 - 12) + 5(2 - 24)$$

$$= -3.$$

5 Find the value of the determinant

$$\Delta = \begin{vmatrix} 8 & 7 & 2 & 20 \\ 3 & 1 & 4 & 7 \\ 5 & 0 & 11 & 0 \\ 8 & 1 & 0 & 6 \end{vmatrix}.$$

Expanding in terms of the third row, since two of the constituents in that row vanish, we have without difficulty

$$\Delta = 5 \begin{vmatrix} 7 & 2 & 20 \\ 1 & 4 & 7 \\ 1 & 0 & 6 \end{vmatrix} + 11 \begin{vmatrix} 8 & 7 & 20 \\ 3 & 1 & 7 \\ 8 & 1 & 6 \end{vmatrix};$$

and expanding the two determinants of the third order, we find $\Delta = 2188$.

6. Expand

$$\begin{vmatrix} 0 & c & b & d \\ c & 0 & a & e \\ b & a & 0 & f \\ d & e & f & 0 \end{vmatrix}.$$

The expansion is $a^2d^2 + b^2e^2 + c^2f^2 - 2bcef - 2cafd - 2abde$; the given determinant is therefore equal to the product of the four factors

$$\sqrt{ad} + \sqrt{be} + \sqrt{cf}, \qquad \sqrt{ad} - \sqrt{be} - \sqrt{cf},$$
$$-\sqrt{ad} + \sqrt{be} - \sqrt{cf}, \qquad -\sqrt{ad} - \sqrt{be} + \sqrt{cf},$$

a result which is sometimes useful.

7. Prove

$$\begin{vmatrix} 1 & \alpha & \beta & \gamma \\ -\alpha & 1 & \gamma' & -\beta' \\ -\beta & -\gamma' & 1 & \alpha' \\ -\gamma & \beta' & -\alpha' & 1 \end{vmatrix} = 1 + \alpha^2 + \beta^2 + \gamma^2 + \alpha'^2 + \beta'^2 + \gamma'^2 + (\alpha\alpha' + \beta\beta' + \gamma\gamma')^2.$$

8. Expand

$$\begin{vmatrix} -a & b & c & d \\ b & -a & d & c \\ c & d & -a & b \\ d & c & b & -a \end{vmatrix}.$$

Ans. $a^4 + b^4 + c^4 + d^4 - 2b^2c^2 - 2c^2a^2 - 2a^2b^2 - 2a^2d^2 - 2b^2d^2 - 2c^2d^2 - 8abcd.$

9. Prove the following identity, and expand the determinants:—

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & z^2 & y^2 \\ 1 & z^2 & 0 & x^2 \\ 1 & y^2 & x^2 & 0 \end{vmatrix} = \begin{vmatrix} 0 & x & y & z \\ x & 0 & z & y \\ y & z & 0 & x \\ z & y & x & 0 \end{vmatrix}.$$

Ans. $x^4 + y^4 + z^4 - 2y^2z^2 - 2z^2x^2 - 2x^2y^2.$

10. Find the value of the determinant

$$\Delta = \begin{vmatrix} a & h & g & \lambda \\ h & b & f & \mu \\ g & f & c & \nu \\ \lambda & \mu & \nu & 0 \end{vmatrix}.$$

Expand first in terms of the last row or last column, and then each of the determinants of the third order in terms of λ, μ, ν.

Ans. $-\Delta = (bc - f^2)\lambda^2 + (ca - g^2)\mu^2 + (ab - h^2)\nu^2 + 2(gh - af)\mu\nu + 2(hf - bg)\nu\lambda + 2(fg - ch)\lambda\mu.$

135. Laplace's Development of a Determinant.—The expansion explained in the preceding Article is included in a more general mode of development given by Laplace. In place of expanding the determinant as a linear function of the constituents of any line, we now expand it as a linear function of the minors comprised in any number of lines.

Consider, for example, the first two columns (a, b) of any determinant; and let all possible determinants of the second order (a_p, b_q), obtained by taking any two rows of these two columns, be formed. Let the minor formed by suppressing the a_p and b_q lines be represented by $\Delta_{p,q}$; then the determinant can be expanded in the form $\Sigma \pm (a_p b_q) \Delta_{p,q}$, where each term is the product of two complementary determinants (see Art. 133). To prove this, we observe that every term of the determinant must contain one constituent from the column a and one from the column b. Suppose a term to contain the factor $a_p b_q$; there must then (interchanging p and q) be another term differing only in the sign and the interchange of these suffixes; hence, the determinant can be expanded in the form $\Sigma (a_p b_q) A_{p,q}$; and $A_{p,q}$ is clearly the sum of all the terms which can be obtained by permuting in every possible way the $n-2$ suffixes of the letters c, d, e, &c., viz. $\pm \Delta_{p,q}$, the sign being determined in any particular instance by the rule of Art. 128. This reasoning can easily be extended to the general case. Let any number p of columns be taken, and all possible minors formed by taking p rows of these columns. Each of these minors is to be then multiplied by the complementary minor, and the determinant expressed as the sum of all such products with their proper signs.

Examples.

1. Expand the determinant $(a_1 b_2 c_3 d_4)$ in terms of the minors of the second order formed from the first two columns.

Employing the bracket notation, we can write down the result as follows:—

$$(a_1 b_2)(c_3 d_4) - (a_1 b_3)(c_2 d_4) + (a_1 b_4)(c_2 d_3) + (a_2 b_3)(c_1 d_4) - (a_2 b_4)(c_1 d_3) + (a_3 b_4)(c_1 d_2);$$

where the sign to be attached to any product is determined by moving the two rows involved in the first factor into the positions of first and second row. Thus, for example, since three displacements are required to move the second and fourth rows into these positions, the sign of the product $(a_3b_4)(c_1d_2)$ is negative.

2. Expand similarly the determinant $(a_1b_2c_3d_4e_5)$.

$$\begin{aligned} \textit{Ans. } & (a_1b_2)(c_3d_4e_5) - (a_1b_3)(c_2d_4e_5) + (a_1b_4)(c_2d_3e_5) - (a_1b_5)(c_2d_3e_4) \\ & + (a_2b_3)(c_1d_4e_5) - (a_2b_4)(c_1d_3e_5) + (a_2b_5)(c_1d_3e_4) + (a_3b_4)(c_1d_2e_5) \\ & - (a_3b_5)(c_1d_2e_4) + (a_4b_5)(c_1d_2e_3). \end{aligned}$$

3. Prove the identity

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \\ 0 & 0 & 0 & \alpha_1 & \beta_1 & \gamma_1 \\ 0 & 0 & 0 & \alpha_2 & \beta_2 & \gamma_2 \\ 0 & 0 & 0 & \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}.$$

This appears by expanding the determinant in terms of the minors formed from the first three columns, for it is evident that all these minors vanish (having one row at least of ciphers) except one, viz. $(a_1b_2c_3)$.

In general, it appears in the same way that if a determinant of the $2m^{th}$ order contains in any position a square of m^2 ciphers, it can be expressed as the product of two determinants of the m^{th} order.

4. Expand the determinant

$$\begin{vmatrix} a & h & g & \lambda & \lambda' \\ h & b & f & \mu & \mu' \\ g & f & c & \nu & \nu' \\ \lambda & \mu & \nu & 0 & 0 \\ \lambda' & \mu' & \nu' & 0 & 0 \end{vmatrix}$$

in powers of α, β, γ, where

$$\alpha \equiv \mu\nu' - \mu'\nu, \quad \beta \equiv \nu\lambda' - \nu'\lambda, \quad \gamma \equiv \lambda\mu' - \lambda'\mu.$$

Ans. $a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta$.

5. Verify the development of the present Article by showing that it gives in the general case the proper number of terms.

Consider the first r columns of a determinant of the n^{th} order. The number of minors formed from these is equal to the number of combinations of n things taken r together. This number multiplied by $1 . 2 . 3 \ldots r$ (the number of terms in each minor), and $1 . 2 . 3 \ldots n - r$ (the number of terms in each complementary minor), will be found to give $1 . 2 . 3 \ldots n$, viz. the number of terms in the determinant.

136. Development of a Determinant in Products of the leading Constituents.—In this and the next following Articles will be explained two additional modes of development which will be found useful in the expansion of certain determinants of special form. The application which follows will be sufficient to show how any determinant may be expanded in products of the leading constituents—

It is required to expand the determinant of the fourth order

$$\Delta \equiv \begin{vmatrix} A & b_1 & c_1 & d_1 \\ a_2 & B & c_2 & d_2 \\ a_3 & b_3 & C & d_3 \\ a_4 & b_4 & c_4 & D \end{vmatrix}$$

according to the products of A, B, C, D. In order to give prominence to the leading constituents, we have here replaced a_1, b_2, c_3, d_4 by A, B, C, D. When the expansion is effected, it is plain that the result must be of the form

$$\Delta \equiv \Delta_0 + \Sigma\lambda A + \Sigma\lambda' AB + ABCD,$$

where Δ_0 consists of all the terms in which no leading constituent occurs; $\Sigma\lambda A$ is the sum of all the terms in which one only of these constituents occurs; $\Sigma\lambda' AB$ is the sum of all in which the product of a pair of the leading constituents is found; and $ABCD$, the leading term, is the product of all these constituents. It will be observed that the expansion here written contains no terms of the form $\lambda'' ABC$; and it is evident, in general, that the expanded determinant can contain no terms in which products of all the leading constituents but one occur, since the coefficient of any such product is the remaining diagonal constituent. It only remains to see what is the form of Δ_0, and of the undetermined coefficients λ, μ, . . . λ', μ', . . . &c.

Putting A, B, C, D all equal to zero in the identity above written, we have

$$\Delta_0 \equiv \begin{vmatrix} 0 & b_1 & c_1 & d_1 \\ a_2 & 0 & c_2 & d_2 \\ a_3 & b_3 & 0 & d_3 \\ a_4 & b_4 & c_4 & 0 \end{vmatrix}.$$

Again, to obtain λ, let B, C, D be made equal to zero. The coefficient of A is clearly the determinant

$$\begin{vmatrix} 0 & c_2 & d_2 \\ b_3 & 0 & d_3 \\ b_4 & c_4 & 0 \end{vmatrix};$$

the coefficient of B is similarly obtained by replacing A, C, D each by zero in the minor complementary to B; and so on. To obtain X', let C and D be made zero; the coefficient of AB in the resulting determinant is plainly the second minor

$$\begin{vmatrix} 0 & d_1 \\ c_4 & 0 \end{vmatrix}.$$

The coefficient of any other product is obtained in a similar manner. Finally, the expansion of Δ may be written in the form

$$\begin{vmatrix} 0 & b_1 & c_1 & d_1 \\ a_2 & 0 & c_2 & d_2 \\ a_3 & b_3 & 0 & d_3 \\ a_4 & b_4 & c_4 & 0 \end{vmatrix}$$

$$+ A\begin{vmatrix} 0 & c_2 & d_2 \\ b_3 & 0 & d_3 \\ b_4 & c_4 & 0 \end{vmatrix} + B\begin{vmatrix} 0 & c_1 & d_1 \\ a_3 & 0 & d_3 \\ a_4 & c_4 & 0 \end{vmatrix} + C\begin{vmatrix} 0 & b_1 & d_1 \\ a_2 & 0 & d_2 \\ a_4 & b_4 & 0 \end{vmatrix} + D\begin{vmatrix} 0 & b_1 & c_1 \\ a_2 & 0 & c_2 \\ a_3 & b_3 & 0 \end{vmatrix}$$

$$+ AB\begin{vmatrix} 0 & d_3 \\ c_4 & 0 \end{vmatrix} + AC\begin{vmatrix} 0 & d_2 \\ b_4 & 0 \end{vmatrix} + AD\begin{vmatrix} 0 & c_2 \\ b_3 & 0 \end{vmatrix} + BC\begin{vmatrix} 0 & d_1 \\ a_4 & 0 \end{vmatrix} + BD\begin{vmatrix} 0 & c_1 \\ a_3 & 0 \end{vmatrix} + CD\begin{vmatrix} 0 & b_1 \\ a_2 & 0 \end{vmatrix}$$

$$+ ABCD.$$

A determinant whose leading constituents all vanish has been called *zero-axial*. The result just obtained may be stated as follows :—*Any determinant may be expanded in products of the leading constituents, the co-factor of every product in the result being a zero-axial determinant.*

137. Expansion of a Determinant in Products in Pairs of the Constituents of a Row and Column.—In what follows we take the first row and first column as those in terms of which the expansion is required. This is evidently sufficient, since any other row and column may be brought by displacements into these positions. It will be found convenient to write the determinant under consideration in the form

$$\begin{vmatrix} a_0 & \alpha & \beta & \gamma & . \\ \alpha' & a_1 & b_1 & c_1 & . \\ \beta' & a_2 & b_2 & c_2 & . \\ \gamma' & a_3 & b_3 & c_3 & . \\ . & . & . & . & . \end{vmatrix}.$$

Let this be denoted by Δ', and its leading first minor $(a_1 b_2 c_3 \ldots)$ by the usual notation Δ. The determinant Δ' may be said to be derived from Δ by *bordering* it, horizontally with the constituents $a_0, \alpha, \beta, \gamma, \ldots$, and vertically with the constituents $a_0, \alpha', \beta', \gamma' \ldots$ When Δ' is expanded, all the terms which contain a_0 are included in $a_0\Delta$. In addition to this, the expansion will consist of the product of every other constituent of the first column by every other constituent of the first row, every such product of two being multiplied by its proper factor. What this factor is in the case of any product is easily seen. Let the co-factors of $a_1, b_1, c_1, \ldots a_2, b_2, \ldots$ &c., in the expansion of Δ be $A_1, B_1, \ldots A_2, B_2, \ldots$, according to the notation explained in Art. 134. It is plain that the factor which multiplies any product, for example $\alpha\alpha'$, in the expansion of Δ', is the same as the factor which multiplies $a_0 a_1$ with sign changed, viz. $-A_1$; similarly the factor which multiplies $\alpha'\beta$ is the factor with sign changed of $a_0 b_1$, viz. $-B_1$; and so on. To obtain the factor of any such product, the rule clearly is:—*Find the fourth constituent completing the rectangle formed by the leading term* a_0 and *the two constituents which enter into the product: the required factor is obtained by substituting for the constituent of* Δ *so found the corresponding capital letter with the negative sign.* It appears therefore finally that the expansion of Δ' may be written in the following form:—

$$
\begin{aligned}
\Delta' \equiv a_0\Delta & - A_1\alpha\alpha' - B_1\beta\alpha' - C_1\gamma\alpha' - \ldots \\
& - A_2\alpha\beta' - B_2\beta\beta' - C_2\gamma\beta' - \ldots \\
& - A_3\alpha\gamma' - B_3\beta\gamma' - C_3\gamma\gamma' - \ldots \\
& - \&c.
\end{aligned}
$$

Examples of the utility of this mode of expansion will be found under a subsequent Article.

138. **Addition of Determinants.** PROP. V.—*If every constituent in any line can be resolved into the sum of two others, the determinant can be resolved into the sum of two others.*

Suppose the constituents of the first column to be $a_1 + \alpha_1$, $a_2 + \alpha_2$, $a_3 + \alpha_3$, &c. Substituting these in the expansion of Art. 134, we have

$$\Delta = (a_1 + \alpha_1) A_1 + (a_2 + \alpha_2) A_2 + (a_3 + \alpha_3) A_3 + \&c.$$

$$= a_1 A_1 + a_2 A_2 + a_3 A_3 + \ldots \&c. + \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \&c.;$$

or,

$$\begin{vmatrix} a_1 + \alpha_1 & b_1 & c_1 & \ldots \\ a_2 + \alpha_2 & b_2 & c_2 & \ldots \\ a_3 + \alpha_3 & b_3 & c_3 & \ldots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 & \ldots \\ a_2 & b_2 & c_2 & \ldots \\ a_3 & b_3 & c_3 & \ldots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 & \ldots \\ \alpha_2 & b_2 & c_2 & \ldots \\ \alpha_3 & b_3 & c_3 & \ldots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

which proves the proposition.

If a second column consists of the sum of two others, it is easily seen, by first resolving with reference to one column, and afterwards with reference to the other, that the determinant can be resolved into the sum of four others. For example, the determinant

$$\begin{vmatrix} a_1 + \alpha_1 & b_1 + \beta_1 & c_1 \\ a_2 + \alpha_2 & b_2 + \beta_3 & c_2 \\ a_3 + \alpha_3 & b_3 + \beta_3 & c_3 \end{vmatrix}$$

is (in the notation of Art. 133) equal to the sum of the four determinants

$$(a_1 b_2 c_3) + (\alpha_1 b_2 c_3) + (a_1 \beta_2 c_3) + (\alpha_1 \beta_2 c_3).$$

Similarly it follows that if each of the constituents of one column consists of the algebraical sum of any number of terms, the determinant can be resolved into a corresponding number of determinants. For example—

$$\begin{vmatrix} a_1 - \alpha_1 + a'_1 & b_1 & c_1 \\ a_2 - \alpha_2 + a'_2 & b_2 & c_2 \\ a_3 - \alpha_3 + a'_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} - \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a'_1 & b_1 & c_1 \\ a'_2 & b_2 & c_2 \\ a'_3 & b_3 & c_3 \end{vmatrix}.$$

And, in general, if one column consists of the algebraic sum of m others, a second column of the sum of n others, a third of the sum of p others, &c., the determinant can be resolved into the sum of $mnp\ldots$, &c., others.

Similar results plainly hold with regard to the rows, which may be substituted for columns in the proof just given.

139. Prop. VI.—*If the constituents of one line are equal to the sums of the corresponding constituents of the other lines multiplied by constant factors, the determinant vanishes.*

For it can then be resolved into the sum of a number of determinants which separately vanish. For example,

$$\begin{vmatrix} ma_1 + nb_1 & a_1 & b_1 \\ ma_2 + nb_2 & a_2 & b_2 \\ ma_3 + nb_3 & a_3 & b_3 \end{vmatrix} = m \begin{vmatrix} a_1 & a_1 & b_1 \\ a_2 & a_2 & b_2 \\ a_3 & a_3 & b_3 \end{vmatrix} + n \begin{vmatrix} b_1 & a_1 & b_1 \\ b_2 & a_2 & b_2 \\ b_3 & a_3 & b_3 \end{vmatrix},$$

and each of the latter determinants vanishes (Art. 130).

140. Prop. VII.—*A determinant is unchanged when to each constituent of any row or column are added those of several other rows or columns multiplied respectively by constant factors.*

For when the determinant is resolved into the sum of others, as in Art. 138, the determinants in which the added lines occur all vanish, since each of them must, when the constant factor is removed, contain two identical lines. Thus, for example,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + mb_1 + nc_1 & b_1 & c_1 \\ a_2 + mb_2 + nc_2 & b_2 & c_2 \\ a_3 + mb_3 + nc_3 & b_3 & c_3 \end{vmatrix};$$

for when the second determinant is expressed as the sum of three others, the two arising from the added columns vanish identically (Art. 139).

The proposition of the present Article supplies in practice one of the most useful properties in the evaluation of determinants.

EXAMPLES.

1. Show that the following determinant vanishes :—

$$\begin{vmatrix} \beta+\gamma & \alpha & 1 \\ \gamma+\alpha & \beta & 1 \\ \alpha+\beta & \gamma & 1 \end{vmatrix}.$$

Adding the constituents of the second column to those of the first, we can take out $\alpha+\beta+\gamma$ as a factor, and two columns then become identical.

2. Find the value of the determinant

$$\begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 3 & 4 & 10 \end{vmatrix}.$$

Subtracting the constituents of the first column from those of the second, and three times the constituents of the first column from those of the third, we obtain

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{vmatrix},$$

which vanishes identically.

3. $$\begin{vmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 2 & 0 \end{vmatrix} = -\begin{vmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{vmatrix} = -16.$$

Here the first transformation is obtained by adding in succession the constituents of the first row to those of the second, third, and fourth.

4. $$\begin{vmatrix} 7 & 11 & 4 \\ 13 & 15 & 10 \\ 3 & 9 & 6 \end{vmatrix} = 3\begin{vmatrix} 7 & 11 & 4 \\ 13 & 15 & 10 \\ 1 & 3 & 2 \end{vmatrix} = 3\begin{vmatrix} 7 & -10 & -10 \\ 13 & -24 & -16 \\ 1 & 0 & 0 \end{vmatrix} = 3\begin{vmatrix} 10 & 10 \\ 24 & 16 \end{vmatrix}$$

$$= 30(16-24) = -240.$$

Here the second transformation is obtained by subtracting three times the first column from the second, and twice the first from the third. In examples of this kind, attempts should be made to reduce to zero all the constituents except one in some row or column, in which case the calculation reduces to that of a determinant of lower order. This can always be done by reducing any one line to units, as in Ex. 7, Art. 132; but, in general, it can be effected more readily by direct additions or subtractions, as in the present instance.

5. $$\begin{vmatrix} 7 & -2 & 0 & 5 \\ -2 & 6 & -2 & 2 \\ 0 & -2 & 5 & 3 \\ 5 & 2 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 7 & -2 & 0 & 5 \\ 19 & 0 & -2 & 17 \\ -7 & 0 & 5 & -2 \\ 12 & 0 & 3 & 9 \end{vmatrix} = 2 \begin{vmatrix} 19 & -2 & 17 \\ -7 & 5 & -2 \\ 12 & 3 & 9 \end{vmatrix}.$$

The first transformation is obtained by adding to the second row three times the first, subtracting the first from the third row, and adding the first to the fourth row. The reduced determinant is easily calculated by subtracting four times the second column from the first, and three times the second column from the third. Thus

$$2 \begin{vmatrix} 19 & -2 & 17 \\ -7 & 5 & -2 \\ 12 & 3 & 9 \end{vmatrix} = 2 \begin{vmatrix} 27 & -2 & 23 \\ -27 & 5 & -17 \\ 0 & 3 & 0 \end{vmatrix} = -6 \begin{vmatrix} 27 & 23 \\ -27 & -17 \end{vmatrix} = -972.$$

6. Calculate the determinant

$$\Delta = \begin{vmatrix} 1 & 15 & 14 & 4 \\ 12 & 6 & 7 & 9 \\ 8 & 10 & 11 & 5 \\ 13 & 3 & 2 & 16 \end{vmatrix}.$$

The first sixteen natural numbers are arranged here in what is called a "magic square," *i.e.* the sum of all the figures in any row or in any column is constant. In general, for a square of the first n^2 numbers, this sum is $\frac{1}{2}n(n^2+1)$. Determinants of this kind can be at once reduced one degree. Here, adding the last three columns to the first, and subtracting the last row from each of the others, we have

$$\Delta = 34 \begin{vmatrix} 1 & 15 & 14 & 4 \\ 1 & 6 & 7 & 9 \\ 1 & 10 & 11 & 5 \\ 1 & 3 & 2 & 16 \end{vmatrix} = 34 \begin{vmatrix} 0 & 12 & 12 & -12 \\ 0 & 3 & 5 & -7 \\ 0 & 7 & 9 & -11 \\ 1 & 3 & 2 & 16 \end{vmatrix} = -34 \times 12 \begin{vmatrix} 1 & 1 & -1 \\ 3 & 5 & -7 \\ 7 & 9 & -11 \end{vmatrix};$$

and subtracting the second row from the last row, it is evident that the reduced determinant vanishes; hence $\Delta = 0$.

7. Calculate the determinant formed by the first nine natural numbers arranged in a magic square:

$$\begin{vmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{vmatrix}.$$

Ans. 360.

8. Calculate the determinant formed by the first twenty-five natural numbers arranged in a magic square:

$$\begin{vmatrix} 10 & 18 & 1 & 14 & 22 \\ 4 & 12 & 25 & 8 & 16 \\ 23 & 6 & 19 & 2 & 15 \\ 17 & 5 & 13 & 21 & 9 \\ 11 & 24 & 7 & 20 & 3 \end{vmatrix}$$

Ans. -4680000.

9. Evaluate, by the method of the present Article, the determinant of Ex. 9, Art. 134.

$$\Delta = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & z^2 & y^2 \\ 1 & z^2 & 0 & x^2 \\ 1 & y^2 & x^2 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & z^2 & y^2 \\ 1 & z^2 & -z^2 & x^2-z^2 \\ 1 & y^2 & x^2-y^2 & -y^2 \end{vmatrix} = -\begin{vmatrix} 1 & z^2 & y^2 \\ 1 & -z^2 & x^2-z^2 \\ 1 & x^2-y^2 & -y^2 \end{vmatrix}.$$

Here, to obtain the second determinant, we subtract the second column from each of the following ones. In the reduced determinant, subtracting the first row from each of the following, we find

$$\Delta = -\begin{vmatrix} 1 & z^2 & y^2 \\ 0 & -2z^2 & x^2-z^2-y^2 \\ 0 & x^2-y^2-z^2 & -2y^2 \end{vmatrix} = -\begin{vmatrix} 2z^2 & y^2+z^2-x^2 \\ y^2+z^2-x^2 & 2y^2 \end{vmatrix}$$

$$= (y^2+z^2-x^2)^2 - 4y^2z^2$$

$$= (y^2+z^2-x^2+2yz)(y^2+z^2-x^2-2yz)$$

$$= \{(y+z)^2-x^2\}\{(y-z)^2-x^2\}$$

$$= -(x+y+z)(y+z-x)(z+x-y)(x+y-z).$$

10. Prove the identity

$$\Delta = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3.$$

Subtracting the last column from each of the others, $(a+b+c)^2$ may be taken out as a factor. Calling the remaining determinant Δ', and subtracting in it the sum of the first two rows from the last, we have

$$\Delta' = \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix} = \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix}$$

$$= \frac{1}{ab}\begin{vmatrix} a(b+c-a) & 0 & a^2 \\ 0 & b(c+a-b) & b^2 \\ -2ab & -2ab & 2ab \end{vmatrix}.$$

Adding the last column to each of the others, we obtain

$$\Delta' = \frac{1}{ab}\begin{vmatrix} a(b+c) & a^2 & a^2 \\ b^2 & b(c+a) & b^2 \\ 0 & 0 & 2ab \end{vmatrix} = 2\begin{vmatrix} a(b+c) & a^2 \\ b^2 & b(c+a) \end{vmatrix} = 2ab\begin{vmatrix} b+c & a \\ b & c+a \end{vmatrix}$$

$$= 2abc(a+b+c).$$

Hence, $\Delta = \Delta'(a+b+c)^2 = 2abc(a+b+c)^3.$

11. Prove the identity

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^3 & \beta^3 & \gamma^3 \end{vmatrix} \equiv (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma).$$

Subtracting the first column from each of the others, $\beta-\alpha$ and $\gamma-\alpha$ become factors. In the reduced determinant, subtract the first row multiplied by α^2 from the second row.

12. Resolve into simple factors the determinant

$$\Delta \equiv \begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^4 & \beta^4 & \gamma^4 & \delta^4 \end{vmatrix}.$$

Proceeding, as in Ex. 11, we easily find that $(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)$ is a factor, and that the reduced determinant is

$$\begin{vmatrix} 1 & 1 & 1 \\ \beta+\alpha & \gamma+\alpha & \delta+\alpha \\ \beta^3+\beta^2\alpha+\beta\alpha^2+\alpha^3 & \gamma^3+\gamma^2\alpha+\gamma\alpha^2+\alpha^3 & \delta^3+\delta^2\alpha+\delta\alpha^2+\alpha^3 \end{vmatrix}.$$

Subtracting the first column from each of the others, $(\gamma-\beta)(\delta-\beta)$ comes out as a factor, and the remaining factor is easily found to be $(\delta-\gamma)(\alpha+\beta+\gamma+\delta)$. Hence, finally,

$$\Delta = -(\beta-\gamma)(\alpha-\delta)(\gamma-\alpha)(\beta-\delta)(\alpha-\beta)(\gamma-\delta)(\alpha+\beta+\gamma+\delta).$$

13. Resolve into linear factors the determinant

$$\Delta \equiv \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}.$$

Multiply the second column by ω, and the third by ω^2; and add to the first. The factor $a+\omega b+\omega^2 c$ may then be taken off the first column (since $\omega^3=1$), leaving the constituents $1, \omega, \omega^2$. Adding then the second and third rows to the first, the factor $a+b+c$ may be taken out; and the remaining determinant is easily found to be equal to $a+\omega^2 b+\omega c$. Hence we have

$$\Delta = (a+b+c)(a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c).$$

14. Resolve into linear factors the determinant

$$\Delta = \begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}.$$

The result is as follows :—

$$\Delta = -(a+b+c+d)(b+c-a-d)(c+a-b-d)(a+b-c-d),$$

since each of the factors here written is a factor of the determinant; for example, $a+b-c-d$ is shown to be a factor by adding the second column to the first, and subtracting the third and fourth. By comparing the sign of a^4, it appears that the negative sign must be attached to the product.

It may be observed that the determinant of Ex. 9 is a particular case of the determinant here considered, viz. that obtained by putting $a = 0$, as will appear by comparing the equivalent forms of Ex. 9, Art. 134.

141. **Multiplication of Determinants.**—Prop. VIII.—*The product of two determinants of any order is itself a determinant of the same order.*

We shall prove this for two determinants of the third order. The student will observe, from the nature of the proof, that it is equally applicable in general. We propose to show that the product of the two determinants $(a_1b_2c_3)$, $(\alpha_1\beta_2\gamma_3)$ is

$$\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix},$$

whose constituents are the sums of the products of the constituents in any row of $(a_1b_2c_3)$ by the corresponding constituents in any row of $(\alpha_1\beta_2\gamma_3)$.

Since each column consists of the sum of three terms, this determinant can be expanded into the sum of twenty-seven others (Art. 138). Now it will be observed that when any one of these is written down, a common factor can be taken off each column; and that several of the partial determinants will, when these factors are removed, have two (or more) columns identical. The determinants which do not vanish in this way can be easily selected. Taking, for example, the first vertical line of the first column, this would give a vanishing determinant if we were to take along with it the first line of the second column. We take then the second line of the second column; and along with these two we must take the third line of the third column to obtain a determinant which does not vanish. Retaining still the first line of the first column, we may take the third line of the second column along with the second line of the third column. Taking out the common factors of the columns, we write down these two determinants as follows:—

$$\alpha_1\beta_2\gamma_3 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \alpha_1\gamma_2\beta_3 \begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix}.$$

Taking in turn each of the other lines of the first column, we obtain four other determinants which do not vanish. Thus there are in all six terms; and it is plain that $(a_1b_2c_3)$ is a factor in each of these. Taking out this factor there remains the sum of six terms—

$$\alpha_1\beta_2\gamma_3 - \alpha_1\beta_3\gamma_2 - \alpha_2\beta_1\gamma_3 + \alpha_3\beta_1\gamma_2 + \alpha_2\beta_3\gamma_1 - \alpha_3\beta_2\gamma_1,$$

and this is the determinant $(\alpha_1\beta_2\gamma_3)$. We have therefore proved that the determinant above written is the product of the two given determinants.

In either of the given determinants the rows may be written in place of columns; hence the product may be written in several different forms as a determinant; but these will, of course, give the same value when expanded.

142. **Multiplication of Determinants continued.**—Another mode of proof of the proposition of the last Article, expressing as a determinant the product of two given determinants of the same order, may be derived from Laplace's mode of development already explained (Art. 135).

The nature of this proof will be sufficiently understood from the application which follows to two determinants of the third order.

The product of the two determinants $(a_1b_2c_3)$, $(\alpha_1\beta_2\gamma_3)$ is (Ex. 3, Art. 135) plainly equal to the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 0 \\ -1 & 0 & 0 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & -1 & 0 & \beta_1 & \beta_2 & \beta_3 \\ 0 & 0 & -1 & \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}$$

In this determinant add to the fourth column the sum of the first multiplied by α_1, the second by β_1, and the third by γ_1; add to the fifth column the sum of the first multiplied by α_2, the second by β_2, and the third by γ_2; and add to the sixth column the sum of the first multiplied by α_3, the second by β_3, and the third by γ_3. The determinant becomes then

$$\begin{vmatrix} a_1 & b_1 & c_1 & a_1\alpha_1+b_1\beta_1+c_1\gamma_1 & a_1\alpha_2+b_1\beta_2+c_1\gamma_2 & a_1\alpha_3+b_1\beta_3+c_1\gamma_3 \\ a_2 & b_2 & c_2 & a_2\alpha_1+b_2\beta_1+c_2\gamma_1 & a_2\alpha_2+b_2\beta_2+c_2\gamma_2 & a_2\alpha_3+b_2\beta_3+c_2\gamma_3 \\ a_3 & b_3 & c_3 & a_3\alpha_1+b_3\beta_1+c_3\gamma_1 & a_3\alpha_2+b_3\beta_2+c_3\gamma_2 & a_3\alpha_3+b_3\beta_3+c_3\gamma_3 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{vmatrix}$$

and this is, by Art. 135, equal to the product (with the proper sign) of the determinant

$$\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} \text{ (which is equal to } -1)$$

by the complementary minor, which is the same determinant as that obtained in the preceding Article. That the sign to be attached to the product is negative is easily seen by moving down the first three rows till the diagonals of the two minors in question form the diagonal of the determinant itself. The student will have no difficulty in observing that, in the general case, the number of such displacements is odd when the order of the given determinants is odd, and even when it is even; so that the sign to be placed before the product-determinant of Art. 141 is always positive.

The important proposition contained in this Article and the Article which precedes will be illustrated by the examples which follow.

Examples.

1. Show that the product of the two determinants

$$\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}, \quad \begin{vmatrix} a'-ib' & c'-id' \\ -c'-id' & a'+ib' \end{vmatrix},$$

where $i=\sqrt{-1}$, may be written in the form

$$\begin{vmatrix} D-iC & B-iA \\ -B-iA & D+iC \end{vmatrix};$$

where

$$A \equiv bc'-b'c+ad'-a'd, \quad B \equiv ca'-c'a+bd'-b'd,$$
$$C \equiv ab'-a'b+cd'-c'd, \quad D \equiv aa'+bb'+cc'+dd';$$

and hence prove Euler's theorem

$$(a^2+b^2+c^2+d^2)(a'^2+b'^2+c'^2+d'^2)$$
$$= (aa'+bb'+cc'+dd')^2+(bc'-b'c+ad'-a'd)^2$$
$$+(ca'-c'a+bd'-b'd)^2+(ab'-a'b+cd'-c'd)^2,$$

viz. *the product of two sums each of four squares can be expressed as the sum of four squares.*

2. Prove the following expression for the square of a determinant of the third order :—

$$2\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}^2 = \begin{vmatrix} 2(ac-b^2) & ac'+a'c-2bb' & ac''+a''c-2bb'' \\ ac'+a'c-2bb' & 2(a'c'-b'^2) & a'c''+a''c'-2b'b'' \\ ac''+a''c-2bb'' & a'c''+a''c'-2b'b'' & 2(a''c''-b''^2) \end{vmatrix}.$$

This appears by multiplying the two determinants

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}, \quad \begin{vmatrix} c & -2b & a \\ c' & -2b' & a' \\ c'' & -2b'' & a'' \end{vmatrix},$$

which differ only by the factor 2.

3. Prove the identity

$$\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} \equiv (a^3+b^3+c^3-3abc)^2.$$

This may be readily proved by multiplying together the two equivalent deter-minants

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}, \quad \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}.$$

4. Prove, by squaring the determinant of Ex. 10, Art. 132, the followin relation between the roots $\alpha, \beta, \gamma, \delta$ of a biquadratic; s_0, s_1, s_2, &c. having th same signification as in Chap. VIII., Vol. I.:—

$$\begin{vmatrix} s_0 & s_1 & s_2 & s_3 \\ s_1 & s_2 & s_3 & s_4 \\ s_2 & s_3 & s_4 & s_5 \\ s_3 & s_4 & s_5 & s_6 \end{vmatrix} = (\beta-\gamma)^2(\alpha-\delta)^2(\gamma-\alpha)^2(\beta-\delta)^2(\alpha-\beta)^2(\gamma-\delta)^2.$$

The student will find no difficulty in writing down for an equation of an degree the corresponding determinant (in terms of the sums of the powers of th roots) which is equal to the product of the squares of the differences.

5. Resolve into factors the determinant

$$\begin{vmatrix} s_6 & s_5 & s_4 & s_3 & x^3 \\ s_5 & s_4 & s_3 & s_2 & x^2 \\ s_4 & s_3 & s_2 & s_1 & x \\ s_3 & s_2 & s_1 & s_0 & 1 \\ y^3 & y^2 & y & 1 & 0 \end{vmatrix},$$

in which s_0, s_1, s_2, &c. are the sums of the powers of three quantities α, β, γ.

This determinant is the product of the two

$$\begin{vmatrix} \alpha^3 & \beta^3 & \gamma^3 & x^3 & 0 \\ \alpha^2 & \beta^2 & \gamma^2 & x^2 & 0 \\ \alpha & \beta & \gamma & x & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}, \quad \begin{vmatrix} \alpha^3 & \beta^3 & \gamma^3 & 0 & y^3 \\ \alpha^2 & \beta^2 & \gamma^2 & 0 & y^2 \\ \alpha & \beta & \gamma & 0 & y \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{vmatrix};$$

and each of the latter can be readily resolved into simple factors.

6. Prove the result of Ex. 28, p. 57, Vol. I., by multiplying the two following determinants:—

$$\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}, \quad \begin{vmatrix} x' & y' & z' \\ z' & x' & y' \\ y' & z' & x' \end{vmatrix}.$$

7. Show that two determinants of different orders may be multiplied together.

For their orders may be made equal; since the order of any determinant can be increased by adding any number of columns and the same number of rows consisting of units in the diagonal, and all the rest zero constituents. For example,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \text{ may be written } \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a_1 & b_1 \\ 0 & 0 & a_2 & b_2 \end{vmatrix},$$

the only effect of the added constituents being to multiply the determinant by unity. More generally, one set of added constituents (*i. e.* those either to the right or the left of the diagonal) might be taken to be any quantities whatever, the remaining set being ciphers. Thus $(a_1 b_2)$ may be written in either of the forms

$$\begin{vmatrix} 1 & \alpha & \beta & \gamma \\ 0 & 1 & \delta & \epsilon \\ 0 & 0 & a_1 & b_1 \\ 0 & 0 & a_2 & b_2 \end{vmatrix}, \quad \begin{vmatrix} 1 & \alpha & \beta & \gamma \\ 0 & 1 & 0 & 0 \\ 0 & \delta & a_1 & b_1 \\ 0 & \epsilon & a_2 & b_2 \end{vmatrix};$$

as readily appears by means of the expansion of Art. 134.

143. Rectangular Arrays.—Arrays in which the number of rows is not equal to the number of columns may be called *rectangular*. These do not themselves represent any definite function; but if two such arrays of the same dimensions are given, there can be derived from them by the process of Art. 141 a determinant whose value we proceed to investigate.

(1). *When the number of columns exceeds the number of rows.*

Take, for example, the two rectangular arrays,

$$\left.\begin{matrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{matrix}\right\} (1), \qquad \left.\begin{matrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \end{matrix}\right\} (2);$$

and performing on these a process similar to that employed in multiplying two determinants, we obtain the determinant

$$\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 + d_1\delta_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 + d_1\delta_2 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 + d_2\delta_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 + d_2\delta_2 \end{vmatrix}.$$

The value of this is easily found to be

$$(a_1b_2)(\alpha_1\beta_2) + (a_1c_2)(\alpha_1\gamma_2) + (a_1d_2)(\alpha_1\delta_2) + (b_1c_2)(\beta_1\gamma_2) \\ + (b_1d_2)(\beta_1\delta_2) + (c_1d_2)(\gamma_1\delta_2),$$

i.e. *the sum of the products of all possible determinants which can be formed from one array (by taking a number of columns equal to the number of rows) multiplied by the corresponding determinants formed from the other array.*

Another proof of this proposition, analogous to the treatment of multiplication of determinants in Art. 142, is given among the examples which follow this Article; and either of these proofs can be easily generalized.

(2). *When the number of rows exceeds the number of columns, the resulting determinant vanishes.*

Take, for example, the two arrays

$$\left.\begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix}\right\} (1), \qquad \left.\begin{matrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{matrix}\right\} (2).$$

Performing the process of multiplication, we have

$$\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 & a_1\alpha_3 + b_1\beta_3 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 & a_2\alpha_3 + b_2\beta_3 \\ a_3\alpha_1 + b_3\beta_1 & a_3\alpha_2 + b_3\beta_2 & a_3\alpha_3 + b_3\beta_3 \end{vmatrix}.$$

It will be observed that this determinant is the same as would arise if a column of ciphers were added to each of the given arrays, and the determinants so formed then multiplied. It follows that the determinant vanishes.

A similar proof applies in general. It is only necessary in any instance to add to each array columns of ciphers, so as to make the number of columns equal to the number of rows, and then multiply the two determinants.

EXAMPLES.

1. From the two arrays

$$\left.\begin{matrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \end{matrix}\right\} (1), \qquad \left.\begin{matrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \end{matrix}\right\} (2),$$

prove

$$\begin{vmatrix} 3 & \alpha+\beta+\gamma \\ \alpha+\beta+\gamma & \alpha^2+\beta^2+\gamma^2 \end{vmatrix} \equiv (\alpha-\beta)^2 + (\alpha-\gamma)^2 + (\beta-\gamma)^2.$$

2. From the two arrays

$$\left.\begin{matrix} a & b & c \\ a' & b' & c' \end{matrix}\right\} (1), \qquad \left.\begin{matrix} c & -2b & a \\ c' & -2b' & a' \end{matrix}\right\} (2),$$

prove

$$4(ac-b^2)(a'c'-b'^2) - (ac'+a'c-2bb')^2 \equiv 4(bc'-b'c)(ab'-a'b) - (ac'-a'c)^2.$$

3. By squaring the array

$$\left.\begin{matrix} a & b & c \\ a' & b' & c' \end{matrix}\right\},$$

prove

$$(a^2+b^2+c^2)(a'^2+b'^2+c'^2) \equiv (aa'+bb'+cc')^2 + (bc'-b'c)^2 + (ca'-c'a)^2 + (ab'-a'b)^2.$$

4. Verify, by squaring the array

$$\left.\begin{matrix} a & b & c & d \\ a' & b' & c' & d' \end{matrix}\right\},$$

the result of Ex. 1, Art. 142.

5. Prove the determinant identity

$$\begin{vmatrix} (a_1-b_1)^2 & (a_1-b_2)^2 & (a_1-b_3)^2 & (a_1-b_4)^2 \\ (a_2-b_1)^2 & (a_2-b_2)^2 & (a_2-b_3)^2 & (a_2-b_4)^2 \\ (a_3-b_1)^2 & (a_3-b_2)^2 & (a_3-b_3)^2 & (a_3-b_4)^2 \\ (a_4-b_1)^2 & (a_4-b_2)^2 & (a_4-b_3)^2 & (a_4-b_4)^2 \end{vmatrix} \equiv 0.$$

This can be proved by multiplying the two arrays

$$\left.\begin{matrix} a_1^2 & a_1 & 1 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_3 & 1 \\ a_4^2 & a_4 & 1 \end{matrix}\right\} (1), \qquad \left.\begin{matrix} 1 & -2b_1 & b_1^2 \\ 1 & -2b_2 & b_2^2 \\ 1 & -2b_3 & b_3^2 \\ 1 & -2b_4 & b_4^2 \end{matrix}\right\} (2).$$

6. For the general equation of the n^{th} degree, whose roots are $\alpha, \beta, \gamma, \delta$, &c., and s_0, s_1, s_2, &c. the sums of the powers of the roots, prove

$$\begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix} = \Sigma(\alpha - \beta)^2.$$

This appears at once by squaring the array

$$\left.\begin{matrix} 1 & 1 & 1 & 1 & 1 & . & . \\ \alpha & \beta & \gamma & \delta & \epsilon & . & . \end{matrix}\right\}.$$

7. Prove similarly, for the general equation,

$$\begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix} = \Sigma(\beta - \gamma)^2(\gamma - \alpha)^2(\alpha - \beta)^2.$$

This is easily proved, as in the preceding example, by squaring a suitable array; and the same process can be used to establish a series of relations of this kind. When the number of rows in the array becomes equal to the degree of the equation, the value of the determinant is the product of the squares of the differences of the roots, as in Ex. 4, Art. 142. When the number of rows exceeds the degree of the equation, the value of the corresponding determinant is zero. The determinant of the fourth order just referred to, for example, vanishes for equations of the second and third degrees.

8. Prove, for the general equation,

$$\begin{vmatrix} s_0 & s_1 & s_2 & s_3 \\ s_1 & s_2 & s_3 & s_4 \\ s_2 & s_3 & s_4 & s_5 \\ 1 & x & x^2 & x^3 \end{vmatrix} = \Sigma(\beta - \gamma)^2(\gamma - \alpha)^2(\alpha - \beta)^2(x - \alpha)(x - \beta)(x - \gamma).$$

Multiplying the two arrays

$$\left.\begin{matrix} 1 & 1 & 1 & . & . \\ \alpha & \beta & \gamma & . & . \\ \alpha^2 & \beta^2 & \gamma^2 & . & . \end{matrix}\right\}, \quad \left.\begin{matrix} x - \alpha & x - \beta & x - \gamma & . & . \\ \alpha(x - \alpha) & \beta(x - \beta) & \gamma(x - \gamma) & . & . \\ \alpha^2(x - \alpha) & \beta^2(x - \beta) & \gamma^2(x - \gamma) & . & . \end{matrix}\right\},$$

we show that Σ is equal to

$$\begin{vmatrix} s_0x - s_1 & s_1x - s_2 & s_2x - s_3 \\ s_1x - s_2 & s_2x - s_3 & s_3x - s_4 \\ s_2x - s_3 & s_3x - s_4 & s_4x - s_5 \end{vmatrix},$$

which is easily transformed into the proposed determinant.

It appears in like manner, in general, that the determinant of similar form of order $p + 1$ is equal to the corresponding symmetric function, each of whose terms contains p factors of the original equation, multiplied by the product of the squared differences of the p roots therein contained.

3. Find the value of the following determinant, and hence derive another proof of the property of arrays of the first kind—

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 & 0 & 0 \\ a_2 & b_2 & c_2 & d_2 & 0 & 0 \\ -1 & 0 & 0 & 0 & \alpha_1 & \alpha_2 \\ 0 & -1 & 0 & 0 & \beta_1 & \beta_2 \\ 0 & 0 & -1 & 0 & \gamma_1 & \gamma_2 \\ 0 & 0 & 0 & -1 & \delta_1 & \delta_2 \end{vmatrix}.$$

Expanding this by Laplace's method, we readily find its value to be the six products, $\Sigma(a_1b_2)(\alpha_1\beta_2)$, of p. 34; and treating the determinant as in Art. 142, viz. adding to the fifth column the sum of the first multiplied by α_1, the second by β_1, &c., we reduce it to the determinant of the second order at the top of p. 34.

144. Solution of a System of Linear Equations.—We have seen in Art. 134 that a determinant may be expanded as a linear homogeneous function of the constituents in any row or column, the coefficient of any constituent being the corresponding minor with its proper sign. We have, for example,

$$\Delta = a_1A_1 + a_2A_2 + a_3A_3 + \&c.$$

Now, the coefficients A_1, A_2, &c., are connected with the constituents of the other columns by $n-1$ identical relations, viz.

$$b_1A_1 + b_2A_2 + b_3A_3 + \&c. = 0,$$

$$c_1A_1 + c_2A_2 + c_3A_3 + \&c. = 0, \&c.;$$

for any one of these is what the determinant becomes when the constituents of the corresponding column are substituted for a_1, a_2, a_3, &c., and must therefore vanish.

By the aid of these relations, we can write down the solution of a system of linear equations. The following application to the case of three unknown quantities x, y, z, is sufficient to explain the general process. Let the equations be

$$a_1x + b_1y + c_1z = m_1,$$

$$a_2x + b_2y + c_2z = m_2,$$

$$a_3x + b_3y + c_3z = m_3.$$

Multiply the first equation by A_1, the second by A_2, and the third by A_3; and add. The coefficients of y and z vanish, in virtue of the relations above proved, and we obtain

$$(a_1A_1 + a_2A_2 + a_3A_3)\,x = m_1A_1 + m_2A_2 + m_3A_3,$$

or

$$\Delta x = \begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix},$$

where Δ represents the determinant formed from the nine constituents a_1, b_1, c_1, &c.

Similarly, multiplying by B_1, B_2, B_3, we obtain

$$(b_1B_1 + b_2B_2 + b_3B_3)\,y = m_1B_1 + m_2B_2 + m_3B_3,$$

$$\Delta y = \begin{vmatrix} a_1 & m_1 & c_1 \\ a_2 & m_2 & c_2 \\ a_3 & m_3 & c_3 \end{vmatrix},$$

where the determinant on the right-hand side is what Δ becomes when m_1, m_2, m_3 are substituted for the constituents of the second column. Similarly, we obtain for z

$$\Delta z = \begin{vmatrix} a_1 & b_1 & m_1 \\ a_2 & b_2 & m_2 \\ a_3 & b_3 & m_3 \end{vmatrix}.$$

These values may be written more compactly as follows:—

$$\Delta x = (m_1b_2c_3), \quad \Delta y = (a_1m_2c_3), \quad \Delta z = (a_1b_2m_3).$$

In general, the values of x, y, z, &c., may be written as follows:—

$$x = \frac{(m_1b_2c_3 \ldots l_n)}{(a_1b_2c_3 \ldots l_n)}, \; y = \frac{(a_1m_2b_3 \ldots l_n)}{(a_1b_2c_3 \ldots l_n)}, \; z = \frac{(a_1b_2m_3 \ldots l_n)}{(a_1b_2c_3 \ldots l_n)}, \text{ \&c.}$$

where, to obtain the value of any unknown, the known quantities m_1, m_2, &c., on the right-hand side of the given equations are to be substituted in Δ for the coefficients of the required unknown, and the determinant so formed to be divided by Δ.

EXAMPLES.

1. Solve the equations

$$x + y + z = A_0,$$
$$\alpha x + \beta y + \gamma z = A_1,$$
$$\alpha^2 x + \beta^2 y + \gamma^2 z = A_2.$$

The solution is easily effected by the formulæ given above. It can be shown that the value of any one of the unknown quantities can be expressed as a quadratic function of its coefficient in these equations, along with *symmetric functions* of α, β, γ (in addition to the given coefficients A_0, A_1, A_2). For this purpose we write the value of the unknown (say, y) in the form

$$\begin{vmatrix} 0 & 1 & 0 & y \\ 1 & 1 & 1 & A_0 \\ \alpha & \beta & \gamma & A_1 \\ \alpha^2 & \beta^2 & \gamma^2 & A_2 \end{vmatrix} = 0, \qquad (1)$$

which may be derived immediately by joining the identical equation $y = y$ to the three given equations, and eliminating after the manner of the Article which follows. Now

$$\begin{vmatrix} 0 & 1 & 0 & y \\ 1 & 1 & 1 & A_0 \\ \alpha & \beta & \gamma & A_1 \\ \alpha^2 & \beta^2 & \gamma^2 & A_2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 & 0 \\ \alpha & \beta & \gamma & 0 \\ \alpha^2 & \beta^2 & \gamma^2 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & \beta & \beta^2 & y \\ s_0 & s_1 & s_2 & A_0 \\ s_1 & s_2 & s_3 & A_1 \\ s_2 & s_3 & s_4 & A_2 \end{vmatrix}.$$

If therefore (assuming that α, β, γ are all unequal), we multiply the equation (1) by the difference-product, we have y expressed as a quadratic function of β along with the sums of the powers of the three quantities α, β, γ.

2. Show, by means of the equations of Art. 77, Vol. I., that the sums of the powers can be expressed in terms of the coefficients, or *vice versâ*, in the form of determinants, as follows:—

$$s_2 = \begin{vmatrix} p_1 & 1 \\ 2p_2 & p_1 \end{vmatrix}, \quad s_3 = -\begin{vmatrix} p_1 & 1 & 0 \\ 2p_2 & p_1 & 1 \\ 3p_3 & p_2 & p_1 \end{vmatrix}, \quad s_4 = \begin{vmatrix} p_1 & 1 & 0 & 0 \\ 2p_2 & p_1 & 1 & 0 \\ 3p_3 & p_2 & p_1 & 1 \\ 4p_4 & p_3 & p_2 & p_1 \end{vmatrix}, \text{ \&c.}$$

$$2p_2 = \begin{vmatrix} s_1 & 1 \\ s_2 & s_1 \end{vmatrix}, \quad 6p_3 = -\begin{vmatrix} s_1 & 1 & 0 \\ s_2 & s_1 & 2 \\ s_3 & s_2 & s_1 \end{vmatrix}, \quad 24p_4 = \begin{vmatrix} s_1 & 1 & 0 & 0 \\ s_2 & s_1 & 2 & 0 \\ s_3 & s_2 & s_1 & 3 \\ s_4 & s_3 & s_2 & s_1 \end{vmatrix}, \text{ \&c.}$$

145. **Linear Homogeneous Equations.**—When $n-1$ linear homogeneous equations between n variables are given, the ratios of the variables can be determined by bringing any one of them to the right-hand side of the equations, and solving as in the previous Article; or we may determine these ratios more conveniently as follows. We take the particular case of three equations between four quantities x, y, z, w, which will be sufficient to illustrate the general process:

$$\left.\begin{aligned} a_1x + b_1y + c_1z + d_1w &= 0 \\ a_2x + b_2y + c_2z + d_2w &= 0 \\ a_3x + b_3y + c_3z + d_3w &= 0 \end{aligned}\right\}. \tag{1}$$

To these may be added a fourth equation whose coefficients are undetermined, viz.

$$a_4x + b_4y + c_4z + d_4w = \lambda. \tag{2}$$

Calling $(a_1b_2c_3d_4)$ as usual Δ, and solving from these four equations by the method of the last Article, we obtain, since $m_1 = 0,\ m_2 = 0,\ m_3 = 0,\ m_4 = \lambda$, the following values:—

$$\Delta x = \lambda A_4, \quad \Delta y = \lambda B_4, \quad \Delta z = \lambda C_4, \quad \Delta w = \lambda D_4,$$

or

$$\frac{x}{A_4} = \frac{y}{B_4} = \frac{z}{C_4} = \frac{w}{D_4} = \frac{\lambda}{\Delta}. \tag{3}$$

The first three of these equations express the ratios of x, y, z, w in terms of the coefficients in the three given equations. And, in general, *the variables are proportional to the coefficients in the expansion of Δ of the constituents of the n^{th} row supposed added to the $n-1$ rows resulting from the given equations.*

We can now express the condition that n linear homogeneous equations should be consistent with one another; for example, that the equation (2) should, when $\lambda = 0$, be consistent with the equations (1). We have only to substitute in (2) the ratios derived from (1), when we obtain

$$a_4A_4 + b_4B_4 + c_4C_4 + d_4D_4 = 0,$$

or

$$\Delta = 0$$

The same thing appears from the equations (3); for if $\lambda = 0$, and if x, y, z, w do not all vanish, Δ must vanish.

What has been proved may be expressed as follows:—*The result of eliminating n quantities between n equations linear and homogeneous in these quantities is the vanishing of the determinant formed by the coefficients of the given equations.*

146. Reciprocal Determinants.—The co-factors $A_1, B_1, C_1 \ldots A_2, B_2$, &c. (Art. 134), which occur in the expansion of a determinant (*i.e.* the first minors with their proper signs), may be called *inverse constituents*; and the determinant formed with them the inverse or *reciprocal determinant*. We proceed to prove certain useful relations connecting the two determinants.

(1). *To express the reciprocal in terms of the given determinant.* Let the reciprocal of Δ be denoted by Δ', and multiply the two determinants

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta' \equiv \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}.$$

All the constituents of the resulting determinant except those in the diagonal vanish (Art. 144); and the result is

$$\Delta\Delta' = \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3;$$

whence $$\Delta' = \Delta^2.$$

The process here employed in the particular case of two determinants of the third order is equally applicable in general; giving $\Delta\Delta' = \Delta^n$, or $\Delta' = \Delta^{n-1}$. Hence *the reciprocal determinant is equal to the $(n-1)^{th}$ power of the given determinant.*

(2). *To express any minor of the reciprocal determinant in terms of the original constituents.*

We take, for example, the determinant of the fourth order, and proceed to express the first minors of its reciprocal. Multiplying the two determinants on the left-hand side of the following equation, and employing the identical equations of Art. 144, we obtain

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 & 0 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 & 0 \\ a_2 & \Delta & 0 & 0 \\ a_3 & 0 & \Delta & 0 \\ a_4 & 0 & 0 & \Delta \end{vmatrix};$$

whence

$$\Delta \begin{vmatrix} B_2 & C_2 & D_2 \\ B_3 & C_3 & D_3 \\ B_4 & C_4 & D_4 \end{vmatrix} = a_1\Delta^3;$$

or $$(B_2C_3D_4) = a_1\Delta^2,$$

thus expressing the first minor of Δ' complementary to A_1.

Again, to express the second minors of Δ', we have, by an exactly similar process,

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 0 & 0 \\ a_2 & b_2 & 0 & 0 \\ a_3 & b_3 & \Delta & 0 \\ a_4 & b_4 & 0 & \Delta \end{vmatrix};$$

whence

$$\Delta \begin{vmatrix} C_3 & D_3 \\ C_4 & D_4 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \Delta^2,$$

or $$(C_3D_4) = (a_1b_2)\,\Delta.$$

The general theorem may be expressed as follows :—*A minor of the order m formed out of the inverse constituents is equal to th complementary of the corresponding minor of the original determi nant Δ multiplied by the $(m-1)^{th}$ power of Δ.*

The method of proof above given can be generalized. In the case of a determinant of the fifth order, for example, the student will easily verify the following expression for a minor of the third order :—

$$(C_3D_4E_5) = (a_1b_2)\,\Delta^2.$$

If the original determinant Δ vanishes, it is plain that not only the reciprocal determinant itself, but also all its minors of any order vanish. The vanishing of the minors of the second order may be expressed in the following useful form :—*When a determinant vanishes, the constituents of any row of its reciprocal are proportional to those of any other row, and the constituents of any column to those of any other column.*

147. Symmetrical Determinants.—Two constituents of a determinant are said to be *conjugate* when one occupies with reference to the leading constituent the same position in the rows as the other does in the columns. For example, d_2 and b_4 are conjugates, one occupying the fourth place in the second row, and the other the fourth place in the second column. Each of the leading constituents is its own conjugate. Any two conjugate constituents are situated in a line perpendicular to the principal diagonal, and at equal distances from it on opposite sides.

A *symmetrical* determinant is one in which every two conjugate constituents are equal to each other. For examples of such determinants the student may refer to Art. 134, Exs. 2, 9, 10, and Art. 135, Ex. 4.

In a symmetrical determinant the first minors complementary to any two conjugate constituents are equal, since they differ only by an interchange of rows and columns. The corresponding inverse constituents are also equal, the signs to be attached to the minors being the same in both cases. It follows that the *reciprocal of a symmetrical determinant is itself symmetrical.*

The leading minors are all symmetrical determinants.

The mode of expansion of Art. 137 is especially useful in the case of symmetrical determinants, as will appear from the examples which follow.

EXAMPLES.

1. Form the reciprocal of the symmetrical determinant

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

Using the capital letters to denote the reciprocal constituents as explained in Art. 134, so that Δ may be expanded in any one of the forms $aA + hH + gG$, $hH + bB + fF$, $gG + fF + cC$, we may write the reciprocal determinant Δ' as follows:—

$$\Delta' \equiv \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} \equiv \begin{vmatrix} bc - f^2 & fg - ch & hf - bg \\ fg - ch & ca - g^2 & gh - af \\ hf - bg & gh - af & ab - h^2 \end{vmatrix}.$$

2. Form similarly the reciprocal of

$$\Delta \equiv \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & d \end{vmatrix}.$$

Using a notation similar to that of the preceding example, so that Δ may be expanded indifferently in any of the forms

$$aA + hH + gG + lL, \quad hH + bB + fF + mM, \ \&c.,$$

the reciprocal determinant Δ' is obtained by replacing in Δ the constituents by the corresponding capital letters. The student will find no difficulty in writing out, if necessary, the expanded form of any of the reciprocal constituents; for example, F is the minor complementary to f with its proper sign (the negative sign in this case), and F is therefore obtained from the expansion of

$$- \begin{vmatrix} a & h & l \\ g & f & n \\ l & m & d \end{vmatrix}.$$

3. Expand the determinant Δ of Ex. 10, Art. 134, by the method of Art. 137. Bringing the last row and last column into the positions of first row and first

column, and using the notation of Ex. 1 for the inverse constituents of the leading minor, the result can be written down at once in the form

$$-\Delta = A\lambda^2 + B\mu^2 + C\nu^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu.$$

Since a determinant is unaltered when both rows and columns are written in reverse order, if the expansion of a determinant be required in terms of the last row and last column (as in the present example), it is not necessary to move them in the first instance into the positions of first row and first column. The expansion can be written down from the determinant as it stands, replacing in the rule of Art. 137 the leading constituent and its minor by the last diagonal constituent and its complementary minor.

4. Expand the determinant Δ of the above Ex. 2, in terms of the last row and column, by the method of Art. 137.

Attending to the remark at the end of the preceding example, and using A, B, C, F, G, H to represent the same quantities as in Exs. 1 and 3, the result may be written down as follows:—

$$\Delta = d\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} - Al^2 - Bm^2 - Cn^2 - 2Fmn - 2Gnl - 2Hlm.$$

When a symmetrical determinant of any order is bordered symmetrically (*i.e.* by the same constituents horizontally and vertically), the result is clearly a symmetrical determinant of the next higher order. The result of Art. 137 shows in general that the expansion of the bordered determinant consists of the original determinant multiplied by the constituent common to the added row and column, together with a homogeneous function of the second degree of the remaining added constituents.

5. Expand the determinant

$$\Delta \equiv \begin{vmatrix} a & h & g & l & \alpha \\ h & b & f & m & \beta \\ g & f & c & n & \gamma \\ l & m & n & d & \delta \\ \alpha & \beta & \gamma & \delta & 0 \end{vmatrix}.$$

This is the determinant of Ex. 2, bordered symmetrically, the common constituent of the added lines being zero. The result is clearly a homogeneous function of the second degree of $\alpha, \beta, \gamma, \delta$; and, by aid of the notation of Ex. 2, the value of $-\Delta$ may be written down at once in the form

$$A\alpha^2 + B\beta^2 + C\gamma^2 + D\delta^2 + 2F\beta\gamma + 2G\gamma\alpha + 2H\alpha\beta + 2L\alpha\delta + 2M\beta\delta + 2N\gamma\delta.$$

6. Prove, by means of the Proposition of Art. 141, that the square of any determinant is a symmetrical determinant.

7. The product of two reciprocal determinants is the reciprocal determinant of the product of the two original determinants.

148. **Skew-Symmetric and Skew Determinants.**—A *skew-symmetric* determinant is one in which every constituent is equal to its conjugate with sign changed. Since each leading constituent is its own conjugate, it follows that in such a determinant all the leading diagonal constituents are zero.

A determinant in which all except the leading constituents are equal to their conjugates with sign changed is called a *skew* determinant. Thus, while a skew-symmetric determinant is zero-axial, in a skew determinant diagonal constituents are present. The calculation of the latter kind may be reduced to that of the former by the method of Art. 136.

The remainder of this article will be occupied with the proof of certain useful properties of skew-symmetric determinants.

(1). *A skew-symmetric determinant of odd order vanishes.*

For any skew-symmetric determinant Δ is unaltered by changing the columns into rows, and then changing the signs of all the rows. But when the order of the determinant is odd, this process ought to change the sign of Δ; hence Δ must in this case vanish. For example,

$$\Delta = \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = 0.$$

(2). *The reciprocal of a skew-symmetric determinant of the n^{th} order is a symmetric determinant when n is odd, and a skew-symmetric determinant when n is even.*

In any skew-symmetric determinant the minors corresponding to a pair of conjugate constituents differ by an interchange of rows and columns, and by the signs of all the constituents. Hence the two minors are equal when their order is even, namely when n is odd; and equal with opposite signs when n is even. In the former case, therefore, the reciprocal determinant is symmetric; and in the latter case it is skew-symmetric, its leading diagonal constituents being all skew-symmetric determinants of odd order.

(3). *A skew-symmetric determinant of even order is a perfect square.*

This follows from the principles established in Art. 146.

Take, for example, the determinant of the fourth order

$$\Delta \equiv \begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix};$$

and let the inverse constituents forming its reciprocal be denoted by $A_1, B_1, \ldots A_2$, &c. We have then, by (2), Art. 146,

$$A_1B_2 - A_2B_1 = \Delta \begin{vmatrix} 0 & f \\ -f & 0 \end{vmatrix} = f^2\Delta.$$

Now A_1 and B_2 being skew-symmetric determinants of odd order, vanish; and $A_2 = -B_1$, since these are conjugate minors; hence $f^2\Delta = A_2^2$, which proves that Δ is a perfect square. Similarly, for a determinant Δ of the sixth order, it is proved that the product of Δ by a skew-symmetric determinant of the fourth order is a perfect square; and since the latter determinant has been just proved to be a perfect square, it follows that Δ is so also. By an exactly similar process, the truth of the proposition having been established for the determinant of the sixth order, it may be proved for one of the eighth; and so on.

EXAMPLES.

1. Verify the following expression for the skew-symmetric determinant of the fourth order:—

$$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix} = (af - be + cd)^2.$$

2. Expand in powers of x the skew determinant

$$\Delta = \begin{vmatrix} x & a & b & c \\ -a & x & d & e \\ -b & -d & x & f \\ -c & -e & -f & x \end{vmatrix}.$$

When the expansion of Art. 136 is employed to calculate a skew determinant, it is to be observed that the determinants of odd order in the expansion all vanish, and those of even order may be expressed as squares. Here the coefficients of the odd powers of x plainly vanish; and the result takes the form

$$\Delta \equiv x^4 + (a^2 + b^2 + c^2 + d^2 + e^2 + f^2)\,x^2 + (af - be + cd)^2.$$

3. Expand the skew determinant

$$\begin{vmatrix} A & a & b & c & d \\ -a & B & e & f & g \\ -b & -e & C & h & i \\ -c & -f & -h & D & j \\ -d & -g & -i & -j & E \end{vmatrix}.$$

The result may be written in the form

$$ABCDE + \Sigma j^2 ABC + \Sigma (ej - fi + gh)^2 A,$$

where the first Σ includes ten terms similar to the one here written, and the second Σ five terms. The terms involving the products in pairs of the leading constituents vanish, as also the term not involving these quantities.

4. The square of any determinant of even order can be expressed as a skew-symmetric determinant.

The following method of proof is applicable in general.

The square of $(a_1 b_2 c_3 d_4)$ is obtained by multiplying the two following determinants:—

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}, \qquad \begin{vmatrix} -b_1 & a_1 & -d_1 & c_1 \\ -b_2 & a_2 & -d_2 & c_2 \\ -b_3 & a_3 & -d_3 & c_3 \\ -b_4 & a_4 & -d_4 & c_4 \end{vmatrix};$$

and the product of these is

$$\begin{vmatrix} 0, & -(a_1b_2)-(c_1d_2), & -(a_1b_3)-(c_1d_3), & -(a_1b_4)-(c_1d_4), \\ (a_1b_2)+(c_1d_2), & 0, & -(a_2b_3)-(c_2d_3), & -(a_2b_4)-(c_2d_4), \\ (a_1b_3)+(c_1d_3), & (a_2b_3)+(c_2d_3), & 0, & -(a_3b_4)-(c_3d_4), \\ (a_1b_4)+(c_1d_4), & (a_2b_4)+(c_2d_4), & (a_3b_4)+(c_3d_4), & 0, \end{vmatrix}$$

which is a skew-symmetric determinant.

5. Form the reciprocal of a skew-symmetric determinant of the third order.

Using for Δ the form in (1) of the present Article, the result is easily found to be the symmetric determinant

$$\begin{vmatrix} c^2 & -bc & ac \\ -bc & b^2 & -ab \\ ac & -ab & a^2 \end{vmatrix}.$$

6. Form the reciprocal of the skew-symmetric determinant Δ of the fourth order in Ex. 1.

Representing by ϕ the function $af - be + cd$ whose square is equal to Δ, and by Δ' the required reciprocal, we easily find

$$\Delta' = \begin{vmatrix} 0 & f\phi & -e\phi & d\phi \\ -f\phi & 0 & c\phi & -b\phi \\ e\phi & -c\phi & 0 & a\phi \\ -d\phi & b\phi & -a\phi & 0 \end{vmatrix}.$$

The value of this skew-symmetric determinant may be written down by aid of the result of Ex. 1. It is thus immediately verified that $\Delta' = (af-be+cd)^2\phi^4 = \Delta^3$.

7. Form the reciprocal of the skew-symmetric determinant $\varDelta$ of the fifth order obtained by making the leading coefficients all vanish in the determinant of Ex. 3.

Since the reciprocal is a symmetric determinant (see (2), Art. 148), and since also it must be such that the constituents of any line are proportional to those of any parallel line (Art. 146), it appears that the required determinant must be of the form

$$\begin{vmatrix} \phi_1^2 & \phi_1\phi_2 & \phi_1\phi_3 & \phi_1\phi_4 & \phi_1\phi_5 \\ \phi_2\phi_1 & \phi_2^2 & \phi_2\phi_3 & \phi_2\phi_4 & \phi_2\phi_5 \\ \phi_3\phi_1 & \phi_3\phi_2 & \phi_3^2 & \phi_3\phi_4 & \phi_3\phi_5 \\ \phi_4\phi_1 & \phi_4\phi_2 & \phi_4\phi_3 & \phi_4^2 & \phi_4\phi_5 \\ \phi_5\phi_1 & \phi_5\phi_2 & \phi_5\phi_3 & \phi_5\phi_4 & \phi_5^2 \end{vmatrix},$$

in which $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5$ are five functions of the second degree in the original constituents whose squares are the values of the five first minors complementary to the leading constituents of Δ.

In general the reciprocal of a skew-symmetric determinant of any odd order $2m+1$ is of a form similar to that just written, the diagonal constituents being the squares, and the remaining constituents the products in pairs, of $2m+1$ functions, each of the m^{th} degree in the original constituents.

149. **Theorem.**—We conclude the present chapter with an important theorem relating to a determinant whose leading first minor vanishes. Adopting the notation of Art. 137, we regard Δ as the vanishing determinant, and state the theorem to be proved as follows: *If a determinant Δ, whose value is zero, be bordered in any manner, the product of the determinant so formed by the leading first minor of Δ is equal to the product of two linear homogeneous functions of the added constituents.*

Retaining the notation of Art. 137, we shall prove that the product of Δ' and A_1 may be expressed in the form:—

$$A_1\Delta' = -(A_1\alpha + B_1\beta + C_1\gamma + \ldots)(A_1\alpha' + A_2\beta' + A_3\gamma' + \ldots).$$

This follows at once from (2) of Art. 146 by considering in the determinant reciprocal to Δ' the values of the constituents inverse to $a_0, \alpha, \alpha', a_1$, and expressing in terms of the original constituents the determinant of the second order formed by these four. Another proof of this result may be readily derived from the expansion of Art. 137, by the aid of the property of the reciprocal of a vanishing determinant (Art. 146), viz., that in the determinant formed by A_1, B_1, C_1, &c., the constituents in any line are proportional to those in any parallel line.

If the determinant Δ is symmetrical, and the bordering also symmetrical, the two factors on the right-hand side of the above equation become identical, and the theorem takes the following form: *If a symmetrical determinant, whose value is zero, be bordered symmetrically, the product of the determinant so formed by its leading second minor is equal to the square with negative sign of a linear homogeneous function of the bordering constituents.*

Regarding Δ' as the original determinant, the following useful statement may be given to the theorem just proved: *If in any symmetrical determinant the leading first minor vanish, the determinant itself and its leading second minor have opposite signs.*

EXAMPLES.

1. If a skew-symmetric determinant Δ of odd order $2m+1$ be bordered in any manner, the resulting determinant Δ' is equal to the product of two rational functions each containing the added constituents in the first degree and the original constituents in the m^{th} degree.

Writing, according to the result of Ex. 7, Art. 148, the reciprocal of the given skew-symmetric determinant in the form

$$\begin{vmatrix} \phi_1^2 & \phi_1\phi_2 & \phi_1\phi_3 & . \\ \phi_2\phi_1 & \phi_2^2 & \phi_2\phi_3 & . \\ . & . & . & . \end{vmatrix},$$

and applying the theorem of the present Article, we find

$$\phi_1^2\Delta' = -(\phi_1^2\alpha + \phi_1\phi_2\beta + \phi_1\phi_3\gamma + \ldots)(\phi_1^2\alpha' + \phi_2\phi_1\beta' + \phi_3\phi_1\gamma' + \ldots),$$
$$\text{or } \Delta' = -(\phi_1\alpha + \phi_2\beta + \phi_3\gamma + \ldots)(\phi_1\alpha' + \phi_2\beta' + \phi_3\gamma' + \ldots).$$

It may be observed that if in this result α', β', γ', &c., be made equal to $-\alpha$, $-\beta$, $-\gamma$, &c., respectively, we fall back on the theorem (3) of Art. 148.

2. If a skew-symmetric determinant of even order $2m$ be bordered in any manner, the resulting determinant is equal to the product of two rational functions, one of the m^{th}, and the other of the $(m+1)^{th}$ degree in the constituents.

This may be derived immediately from the last example by making therein all the added constituents in the first column, viz., α', β', γ', &c., equal to zero, except the last, which is to be made $= 1$. The determinant then reduces to one of the kind here considered, the bordering constituents forming the top row and the last column. It appears also that the factor of the m^{th} degree in the result is the square root of the given skew-symmetric determinant of order $2m$.

3. Prove

$$\begin{vmatrix} 0 & \alpha & \beta & \gamma \\ \alpha' & 0 & c & -b \\ \beta' & -c & 0 & a \\ \gamma' & b & -a & 0 \end{vmatrix} \equiv -(a\alpha + b\beta + c\gamma)(a\alpha' + b\beta' + c\gamma').$$

4. Resolve into its factors

$$\begin{vmatrix} 0 & \alpha & \beta & \gamma & \delta \\ \alpha' & 0 & c & -b & x \\ \beta' & -c & 0 & a & y \\ \gamma' & b & -a & 0 & z \\ \delta' & -x & -y & -z & 0 \end{vmatrix}.$$

Ans. $(ax + by + cz)\{x(\beta\gamma') + y(\gamma\alpha') + z(\alpha\beta') + a(\alpha\delta') + b(\beta\delta') + c(\gamma\delta')\}$.

Miscellaneous Examples.

1. Prove

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} \equiv J,$$

where J has the usual signification.

2. Prove

$$\begin{vmatrix} \beta+\gamma & \gamma+\alpha & \alpha+\beta \\ \beta'+\gamma' & \gamma'+\alpha' & \alpha'+\beta' \\ \beta''+\gamma'' & \gamma''+\alpha'' & \alpha''+\beta'' \end{vmatrix} \equiv 2 \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix}.$$

3. Prove

$$\begin{vmatrix} \beta\gamma & \beta\gamma'+\beta'\gamma & \beta'\gamma' \\ \gamma\alpha & \gamma\alpha'+\gamma'\alpha & \gamma'\alpha' \\ \alpha\beta & \alpha\beta'+\alpha'\beta & \alpha'\beta' \end{vmatrix} \equiv (\beta\gamma')(\gamma\alpha')(\alpha\beta'),$$

where the factors on the right-hand side are determinants of the second order.

Dividing the rows by $\beta'\gamma'$, $\gamma'\alpha'$, $\alpha'\beta'$; and putting $\lambda = \frac{\alpha}{\alpha'}$, $\mu = \frac{\beta}{\beta'}$, $\nu = \frac{\gamma}{\gamma'}$, the determinant (omitting a factor) reduces to the form

$$\begin{vmatrix} 1 & \mu+\nu & \mu\nu \\ 1 & \nu+\lambda & \nu\lambda \\ 1 & \lambda+\mu & \lambda\mu \end{vmatrix} \equiv \begin{vmatrix} 1 & -\lambda & \mu\nu \\ 1 & -\mu & \nu\lambda \\ 1 & -\nu & \lambda\mu \end{vmatrix} \equiv -(\mu-\nu)(\nu-\lambda)(\lambda-\mu), \text{ \&c.}$$

4. Find the value of the determinant

$$\begin{vmatrix} 1 & \beta+\gamma+\delta & \beta\gamma+\beta\delta+\gamma\delta & \beta\gamma\delta \\ 1 & \alpha+\gamma+\delta & \alpha\gamma+\alpha\delta+\gamma\delta & \alpha\gamma\delta \\ 1 & \alpha+\beta+\delta & \alpha\beta+\alpha\delta+\beta\delta & \alpha\beta\delta \\ 1 & \alpha+\beta+\gamma & \alpha\beta+\alpha\gamma+\beta\gamma & \alpha\beta\gamma \end{vmatrix}.$$

Since the interchange of two letters would make two rows identical, this can differ by a numerical factor only from the product of the six differences. Or we may reduce the determinant easily to the form in Ex. 10, Art. 132. The value of a similar determinant of any order can be found in the same way; and the sign can be determined in any instance by the method of Ex. 9, Art. 132.

5. Prove

$$\begin{vmatrix} \beta^2\gamma^2 + \alpha^2\delta^2 & \beta\gamma + \alpha\delta & 1 \\ \gamma^2\alpha^2 + \beta^2\delta^2 & \gamma\alpha + \beta\delta & 1 \\ \alpha^2\beta^2 + \gamma^2\delta^2 & \alpha\beta + \gamma\delta & 1 \end{vmatrix} \equiv (\beta - \gamma)(\alpha - \delta)(\gamma - \alpha)(\beta - \delta)(\alpha - \beta)(\gamma - \delta).$$

Add the last column multiplied by $2\alpha\beta\gamma\delta$ to the first. The determinant becomes then of the form of Ex. 9, Art. 132.

6. Prove

$$\begin{vmatrix} (\beta + \gamma - \alpha - \delta)^4 & (\beta + \gamma - \alpha - \delta)^2 & 1 \\ (\gamma + \alpha - \beta - \delta)^4 & (\gamma + \alpha - \beta - \delta)^2 & 1 \\ (\alpha + \beta - \gamma - \delta)^4 & (\alpha + \beta - \gamma - \delta)^2 & 1 \end{vmatrix} \equiv 64(\beta - \gamma)(\alpha - \delta)(\gamma - \alpha)(\beta - \delta)(\alpha - \beta)(\gamma - \delta).$$

7. Prove

$$\begin{vmatrix} a & b & ax + b \\ b & c & bx + c \\ ax + b & bx + c & 0 \end{vmatrix} \equiv -(ac - b^2)(ax^2 + 2bx + c).$$

Subtract from the third row the second row plus the first multiplied by x.

8. Prove similarly

$$\begin{vmatrix} a & b & c & ax^2 + 2bx + c \\ b & c & d & bx^2 + 2cx + d \\ c & d & e & cx^2 + 2dx + e \\ ax^2+2bx+c & bx^2+2cx+d & cx^2+2dx+e & 0 \end{vmatrix}$$

$$\equiv - \begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix} (ax^4 + 4bx^3 + 6cx^2 + 4dx + e).$$

9. Given

$$f_1(x) = a_1x^3 + 3b_1x^2 + 3c_1x + d_1,$$

$$f_2(x) = a_2x^3 + 3b_2x^2 + 3c_2x + d_2,$$

$$f_3(x) = a_3x^3 + 3b_3x^2 + 3c_3x + d_3;$$

prove the identity

$$\begin{vmatrix} f_1(x) & f_1'(x) & f_1''(x) \\ f_2(x) & f_2'(x) & f_2''(x) \\ f_3(x) & f_3'(x) & f_3''(x) \end{vmatrix} \equiv -18 \begin{vmatrix} 1 & -x & x^2 & -x^3 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}.$$

The first determinant reduces easily (omitting a factor) to the following :—

$$\begin{vmatrix} a_1x + b_1 & b_1x + c_1 & c_1x + d_1 \\ a_2x + b_2 & b_2x + c_2 & c_2x + d_2 \\ a_3x + b_3 & b_3x + c_3 & c_3x + d_3 \end{vmatrix}.$$

We have seen (Ex. 7, Art. 142) that the order of a determinant may be increased without altering its value. By a suitable selection of the added constituents the calculation of a determinant may often be simplified by bordering it in this way. The determinant last written is plainly equal to

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ a_1 & a_1x + b_1 & b_1x + c_1 & c_1x + d_1 \\ a_2 & a_2x + b_2 & b_2x + c_2 & c_2x + d_2 \\ a_3 & a_3x + b_3 & b_3x + c_3 & c_3x + d_3 \end{vmatrix}.$$

Subtracting from the second column the first multiplied by x; subtracting then from the third the new second column multiplied by x; and, finally, from the fourth the new third column multiplied by x, we have the result above stated.

10. Show that the determinant

$$\begin{vmatrix} \lambda x^2 + cy^2 + bz^2 - 1 & (\lambda - c)xy & (\lambda - b)xz \\ (\lambda - c)xy & \lambda y^2 + az^2 + cx^2 - 1 & (\lambda - a)yz \\ (\lambda - b)xz & (\lambda - a)yz & \lambda z^2 + bx^2 + ay^2 - 1 \end{vmatrix}$$

contains $\lambda(x^2 + y^2 + z^2) - 1$ as a factor, and that the remaining factor is independent of λ.

Border the determinant, as in Ex. 9, with a first column whose constituents are $1, \lambda x, \lambda y, \lambda z$; and with a first row whose constituents are $1, 0, 0, 0$. Subtract then x times the first column from the second, y times the first column from the third, and z times the first column from the fourth. In the determinant thus altered, subtract from the first row x times the second plus y times the third plus z times the fourth.

11. Expand in powers of x the determinant

$$\begin{vmatrix} a_1 + x & b_1 & c_1 & d_1 \\ a_2 & b_2 + x & c_2 & d_2 \\ a_3 & b_3 & c_3 + x & d_3 \\ a_4 & b_4 & c_4 & d_4 + x \end{vmatrix}.$$

Ans. $x^4 + (a_1 + b_2 + c_3 + d_4)x^3 + \{(b_2c_3) + (a_1d_4) + (a_1c_3) + (b_2d_4) + (a_1b_2) + (c_3d_4)\}x^2 + \{(b_2c_3d_4) + (a_1c_3d_4) + (a_1b_2d_4) + (a_1b_2c_3)\}x + (a_1b_2c_3d_4).$

12. Prove

$$\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ \frac{a^2}{a'} & \frac{b^2}{b'} & \frac{c^2}{c'} & \frac{d^2}{d'} \\ \frac{a'^2}{a} & \frac{b'^2}{b} & \frac{c'^2}{c} & \frac{d'^2}{d} \end{vmatrix} = \frac{-(bc')(ad')(ca')(bd')(ab')(cd')}{abcda'b'c'd'}.$$

13. Prove the identities

$$\begin{vmatrix} 1 & \alpha & \alpha' & \alpha\alpha' \\ 1 & \beta & \beta' & \beta\beta' \\ 1 & \gamma & \gamma' & \gamma\gamma' \\ 1 & \delta & \delta' & \delta\delta' \end{vmatrix} = \begin{vmatrix} B & C \\ B' & C' \end{vmatrix} = \begin{vmatrix} C & A \\ C' & A' \end{vmatrix} = \begin{vmatrix} A & B \\ A' & B' \end{vmatrix},$$

where

$$A = (\beta - \gamma)(\alpha - \delta), \quad B = (\gamma - \alpha)(\beta - \delta), \quad C = (\alpha - \beta)(\gamma - \delta),$$
$$A' = (\beta' - \gamma')(\alpha' - \delta'), \quad B' = (\gamma' - \alpha')(\beta' - \delta'), \quad C' = (\alpha' - \beta')(\gamma' - \delta').$$

Expanding the first determinant in terms of the minors formed from the first two columns (see Art. 135), we easily prove that it is equal to

$$A(\beta'\gamma' + \alpha'\delta') + B(\gamma'\alpha' + \beta'\delta) + C(\alpha'\beta' + \gamma'\delta');$$

and employing the identical equation $A + B + C = 0$, along with the relations of Ex. 18, Art. 27, the result follows.

14. Prove that the determinant of Ex. 13 is equal to

$$\begin{vmatrix} 1 & \beta\gamma + \alpha\delta & \beta'\gamma' + \alpha'\delta' \\ 1 & \gamma\alpha + \beta\delta & \gamma'\alpha' + \beta'\delta' \\ 1 & \alpha\beta + \gamma\delta & \alpha'\beta' + \gamma'\delta' \end{vmatrix}$$

This follows at once from the relations of Ex. 18, Art. 27. If α', β', γ', δ' be put equal to α^m, β^m, γ^m, δ^m in the result, we obtain an identity which includes Ex. 5, p. 53, as a particular case.

15. Express as a function of differences the following determinant, whose vanishing expresses the condition for involution of six points on a line :—

$$\Delta \equiv \begin{vmatrix} 1 & \alpha + \alpha' & \alpha\alpha' \\ 1 & \beta + \beta' & \beta\beta' \\ 1 & \gamma + \gamma' & \gamma\gamma' \end{vmatrix}.$$

Multiplying the determinant by

$$\begin{vmatrix} \alpha^2 & -\alpha & 1 \\ \beta^2 & -\beta & 1 \\ \gamma^2 & -\gamma & 1 \end{vmatrix},$$

and then removing the factor $(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)$ from both sides of the equation, the value of Δ is easily expressed as follows :—

$$\Delta \equiv (\alpha-\beta')(\beta-\gamma')(\gamma-\alpha')+(\alpha'-\beta)(\beta'-\gamma)(\gamma'-\alpha).$$

This result may also be derived from the determinant of Ex. 13, whose vanishing expresses the general homographic relation between two sets of four points.

16. Expand the determinant

$$\begin{vmatrix} x & 0 & 0 & 0 & a_4 \\ -1 & x & 0 & 0 & a_3 \\ 0 & -1 & x & 0 & a_2 \\ 0 & 0 & -1 & x & a_1 \\ 0 & 0 & 0 & -1 & a_0 \end{vmatrix}.$$

This is found to be identical with the quartic

$$a_0x^4+a_1x^3+a_2x^2+a_3x+a_4;$$

and it is easily seen that a polynomial of any degree can be expressed as a determinant of like form.

17. Prove

$$\begin{vmatrix} x & a_1 & a_2 & a_3 & 1 \\ \alpha & x & b_1 & b_2 & 1 \\ \alpha & \beta & x & c_1 & 1 \\ \alpha & \beta & \gamma & x & 1 \\ \alpha & \beta & \gamma & \delta & 1 \end{vmatrix} \equiv (x-\alpha)(x-\beta)(x-\gamma)(x-\delta);$$

$a_1, a_2, a_3, b_1, b_2, c_1$ being any quantities.

This follows by subtracting α times the last column from the first, β times the last from the second, &c. The student will have no difficulty in writing down the corresponding determinant of the $(n+1)^{th}$ order which is equal to the polynomial $f(x)$ whose roots are $a_1, a_2, a_3 \ldots a_n$.

18. Resolve into factors the determinant

$$\Delta \equiv \begin{vmatrix} (\alpha-\alpha')^2 & (\alpha-\beta')^2 & (\alpha-\gamma')^2 \\ (\beta-\alpha')^2 & (\beta-\beta')^2 & (\beta-\gamma')^2 \\ (\gamma-\alpha')^2 & (\gamma-\beta')^2 & (\gamma-\gamma')^2 \end{vmatrix}.$$

Here
$$\Delta = \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix} \begin{vmatrix} 1 & -2\alpha' & \alpha'^2 \\ 1 & -2\beta' & \beta'^2 \\ 1 & -2\gamma' & \gamma'^2 \end{vmatrix};$$

and these two determinants may be resolved as in Ex. 9, Art. 132.

19. Resolve into factors the determinant

$$\Delta \equiv \begin{vmatrix} (\alpha-\alpha')^3 & (\alpha-\beta')^3 & (\alpha-\gamma')^3 \\ (\beta-\alpha')^3 & (\beta-\beta')^3 & (\beta-\gamma')^3 \\ (\gamma-\alpha')^3 & (\gamma-\beta')^3 & (\gamma-\gamma')^3 \end{vmatrix}.$$

Multiplying the two rectangular arrays

$$\left.\begin{matrix} \alpha^3 & \alpha^2 & \alpha & 1 \\ \beta^3 & \beta^2 & \beta & 1 \\ \gamma^3 & \gamma^2 & \gamma & 1 \end{matrix}\right\} (1), \qquad \left.\begin{matrix} 1 & -3\alpha' & 3\alpha'^2 & -\alpha'^3 \\ 1 & -3\beta' & 3\beta'^2 & -\beta'^3 \\ 1 & -3\gamma' & 3\gamma'^2 & -\gamma'^3 \end{matrix}\right\} (2),$$

Δ becomes equal to the sum of four terms, from each of which we can take out as a factor the product of the two determinants

$$\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix}, \qquad \begin{vmatrix} 1 & \alpha' & \alpha'^2 \\ 1 & \beta' & \beta'^2 \\ 1 & \gamma' & \gamma'^2 \end{vmatrix}.$$

The remaining factor is

$$3\{3\alpha\beta\gamma - \Sigma\beta\gamma\Sigma\alpha' + \Sigma\beta'\gamma'\Sigma\alpha - 3\alpha'\beta'\gamma'\},$$

which can be written also in the form

$$3\{(\alpha-\alpha')(\beta-\beta')(\gamma-\gamma') + (\alpha-\beta')(\beta-\gamma')(\gamma-\alpha') + (\alpha-\gamma')(\beta-\alpha')(\gamma-\beta')\}.$$

20. Prove the expansion

$$\begin{vmatrix} 1+a_1 & 1 & 1 & 1 \\ 1 & 1+a_2 & 1 & 1 \\ 1 & 1 & 1+a_3 & 1 \\ 1 & 1 & 1 & 1+a_4 \end{vmatrix} = a_1a_2a_3a_4\left\{1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\frac{1}{a_4}\right\}.$$

This is easily proved by subtracting the first column from each of the others, and then expanding the determinant as a linear function of the constituents of the first column. It will be apparent from the nature of the proof that the value of the similar determinant of the n^{th} order is $a_1a_2a_3 \ldots a_n \left\{1+\Sigma\frac{1}{a_1}\right\}$.

21. Prove the relation

$$\begin{vmatrix} \alpha & x & x & x \\ x & \beta & x & x \\ x & x & \gamma & x \\ x & x & x & \delta \end{vmatrix} = f(x) - xf'(x).$$

where $\qquad f(x) \equiv (x-\alpha)(x-\beta)(x-\gamma)(x-\delta).$

This can be derived from the preceding example, or proved independently in a similar way. As in the last example, the determinant of this form of the n^{th} degree can be similarly expressed.

22. Each of the coefficients of any equation can be expressed in terms of the roots as the quotient of two determinants.

The student can easily extend to any degree the following application to the equation of the third degree.

From Ex. 10, Art. 132, we have

$$\begin{vmatrix} x^3 & x^2 & x & 1 \\ \alpha^3 & \alpha^2 & \alpha & 1 \\ \beta^3 & \beta^2 & \beta & 1 \\ \gamma^3 & \gamma^2 & \gamma & 1 \end{vmatrix} \equiv -(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(x-\alpha)(x-\beta)(x-\gamma).$$

Expanding the determinant, this identity can be written

$$\begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix} x^3 - \begin{vmatrix} \alpha^3 & \alpha & 1 \\ \beta^3 & \beta & 1 \\ \gamma^3 & \gamma & 1 \end{vmatrix} x^2 + \begin{vmatrix} \alpha^3 & \alpha^2 & 1 \\ \beta^3 & \beta^2 & 1 \\ \gamma^3 & \gamma^2 & 1 \end{vmatrix} x - \begin{vmatrix} \alpha^3 & \alpha^2 & \alpha \\ \beta^3 & \beta^2 & \beta \\ \gamma^3 & \gamma^2 & \gamma \end{vmatrix}$$

$$\equiv \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix} \{x^3 - p_1x^2 + p_2x - p_3\},$$

from which the above proposition follows, p_1, p_2, p_3 being the coefficients of the equation whose roots are α, β, γ.

23. Express as a determinant the reducing cubic of a biquadratic.

Writing down the equations which result from the identity

$$(a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4) \equiv (ax^2 + 2bx + c)(a'x^2 + 2b'x + c'),$$

assuming $6a_0\phi \equiv ac' + a'c - 2bb'$, and substituting in the following identity :—

$$\begin{vmatrix} a & a' & 0 \\ b & b' & 0 \\ c & c' & 0 \end{vmatrix} \times \begin{vmatrix} a' & a & 0 \\ b' & b & 0 \\ c' & c & 0 \end{vmatrix} \equiv \begin{vmatrix} 2aa' & ab' + a'b & ac' + a'c \\ ab' + a'b & 2bb' & bc' + b'c \\ ac' + a'c & bc' + b'c & 2cc' \end{vmatrix} = 0,$$

we easily find the equation

$$\begin{vmatrix} a_0 & a_1 & a_2 + 2a_0\phi \\ a_1 & a_2 - a_0\phi & a_3 \\ a_2 + 2a_0\phi & a_3 & a_4 \end{vmatrix} = 0,$$

which when expanded is found to be identical with the standard reducing cubic.

24. Find the condition that a biquadratic should be capable of being expressed as the sum of two fourth powers; and, expressing it in the form

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e \equiv l(x+\theta)^4 + m(x+\phi)^4,$$

find the quadratic whose roots are θ and ϕ.

From this identity we have the following equations:—

$$\left.\begin{aligned} l + m &= a,\\ l\theta + m\phi &= b,\\ l\theta^2 + m\phi^2 &= c,\\ l\theta^3 + m\phi^3 &= d,\\ l\theta^4 + m\phi^4 &= e,\end{aligned}\right\} (1).$$

Assuming $\lambda + \mu x + \nu x^2 = 0$ as the equation whose roots are θ and ϕ, we easily obtain the three equations,

$$\lambda a + \mu b + \nu c = 0,$$
$$\lambda b + \mu c + \nu d = 0,$$
$$\lambda c + \mu d + \nu e = 0,$$

from which we have at once the required condition $J = 0$; and from the first two, along with the assumed equation, we obtain the following quadratic whose roots are θ and ϕ:—

$$\begin{vmatrix} 1 & x & x^2 \\ a & b & c \\ b & c & d \end{vmatrix} = 0.$$

If it were required to express a cubic as the sum of two cubes, in the form $l(x+\theta)^3 + m(x+\phi)^3$, the first four of the above equations (1) would lead to the same quadratic for θ and ϕ.

25. For the biquadratic

$$A(x+\alpha)^4 + B(x+\beta)^4 + C(x+\gamma)^4 + D(x+\delta)^4 = 0,$$

prove

$$H = \Sigma AB(\alpha-\beta)^2,$$
$$I = \Sigma AB(\alpha-\beta)^4,$$
$$J = \Sigma ABC(\alpha-\beta)^2(\alpha-\gamma)^2(\beta-\gamma)^2.$$

These expressions are true for a biquadratic written as the sum of any number of fourth powers. If it can be written as the sum of two only, $J = 0$, since only A and B remain; and if it reduces to one fourth power, H, I, J all vanish—results already obtained by other methods.

26. Discuss the determinant of the fourth order, whose constituents $(\alpha-\alpha')^4$, $(\alpha-\beta')^4$, &c., are arranged as in Ex. 19, p. 57; and if $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta'$ are the roots of two given biquadratic equations, show that the value in terms of the coefficients contains as a factor

$$ae' + a'e - 4(bd' + b'd) + 6cc'.$$

When the two biquadratics are identical this factor becomes $2I$.

27. Find the condition that the homogeneous quadratic function of three variables

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

should be resolvable into two factors.

Equating the given function to the product of the factors

$$(\alpha x + \beta y + \gamma z)(\alpha' x + \beta' y + \gamma' z),$$

we readily find

$$\begin{vmatrix} \alpha & \alpha' & 0 \\ \beta & \beta' & 0 \\ \gamma & \gamma' & 0 \end{vmatrix} \begin{vmatrix} \alpha' & \alpha & 0 \\ \beta' & \beta & 0 \\ \gamma' & \gamma & 0 \end{vmatrix} = 8 \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix};$$

hence the required condition is that the determinant last written should vanish.

28. Show that the most general values of x, y, z, w which satisfy the two homogeneous equations

$$ax + by + cz + dw = 0, \qquad a'x + b'y + c'z + d'w = 0,$$

may be expressed symmetrically in terms of two indeterminates X, Y in the form

$$(ab')(ac')(ad')\,x = aX + a'Y,$$
$$(ba')(bc')(bd')\,y = bX + b'Y, \text{ \&c.}$$

This may be proved by joining to the two given equations the two following :—

$$\frac{a^2}{a'}x + \frac{b^2}{b'}y + \frac{c^2}{c'}z + \frac{d^2}{d'}w = \lambda, \quad \frac{a'^2}{a}x + \frac{b'^2}{b}y + \frac{c'^2}{c}z + \frac{d'^2}{d}w = \mu,$$

where λ, μ are indeterminate quantities; by then solving for x, y, z, w, as in Art. 144, and reducing the determinants as in Ex. 12, p. 55; and finally making $X = a'b'c'd'\lambda$, $Y = abcd\mu$.

29. If in any determinant r columns (or rows) become identical when $x = a$, then $(x - a)^{r-1}$ is a factor in the determinant.

This appears easily by subtracting in the given determinant one of the r columns from each of the others. The resulting $r - 1$ columns must each contain $x - a$ as a factor, since by hypothesis each constituent in it vanishes when $x = a$.

30. Find the value of the determinant of the n^{th} order

$$\Delta \equiv \begin{vmatrix} x & a & a & . & a \\ a & x & a & . & a \\ a & a & x & . & a \\ . & . & . & . & . \\ a & a & a & . & x \end{vmatrix},$$

whose leading constituents are all equal to x, and the remaining constituents all equal to a.

By the preceding example Δ must contain $(x-a)^{n-1}$ as a factor; and by adding all the columns we see that it must also contain $x+(n-1)a$ as a factor. Hence Δ can differ by a numerical factor only from the product of these; and by comparing the product with the leading term we find

$$\Delta = (x-a)^{n-1}\{x+(n-1)a\}.$$

This result can readily be proved directly without the aid of Ex. 29.

31. The determinant

$$\begin{vmatrix} f_1(\alpha) & f_2(\alpha) & f_3(\alpha) \\ f_1(\beta) & f_2(\beta) & f_3(\beta) \\ f_1(\gamma) & f_2(\gamma) & f_3(\gamma) \end{vmatrix},$$

in which f_1, f_2, f_3 are any rational integral functions, contains the difference-product $(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)$ as a factor.

This appears readily by reasoning similar to that of Ex. 29. Determinants of this nature, in which the constituents of any column (or row) are functions of the same form, and the constituents of any row (or column) involve the same quantity, are called *alternants*. It is clear that the result is general, and that the alternant of any order contains as a factor the difference-product of all the quantities involved. The determinants of Exs. 9, 10, Art. 132, and Exs. 11, 12, Art. 140, are alternants of the simplest form.

32. Express in the form of a determinant the quotient of the alternant in the preceding example by the difference-product.

Assuming, to fix the ideas, that the functions involved are each of the fifth degree (which will include lower degrees by making some coefficients vanish), we may write

$$f_1(\alpha) \equiv a_1\alpha^5 + b_1\alpha^4 + c_1\alpha^3 + d_1\alpha^2 + e_1\alpha + f_1,$$
$$f_2(\alpha) \equiv a_2\alpha^5 + b_2\alpha^4 + c_2\alpha^3 + d_2\alpha^2 + e_2\alpha + f_2,$$
$$f_3(\alpha) \equiv a_3\alpha^5 + b_3\alpha^4 + c_3\alpha^3 + d_3\alpha^2 + e_3\alpha + f_3.$$

Now, taking α, β, γ to be the roots of the equation

$$x^3 + px^2 + qx + r = 0,$$

and forming the product of the following determinants :—

$$\begin{vmatrix} \alpha^5 & \alpha^4 & \alpha^3 & \alpha^2 & \alpha & 1 \\ \beta^5 & \beta^4 & \beta^3 & \beta^2 & \beta & 1 \\ \gamma^5 & \gamma^4 & \gamma^3 & \gamma^2 & \gamma & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}, \quad \begin{vmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 \\ 0 & 0 & 1 & p & q & r \\ 0 & 1 & p & q & r & 0 \\ 1 & p & q & r & 0 & 0 \end{vmatrix},$$

it readily appears that the determinant last written is the required quotient.

A similar method may be used to form the quotient when the alternant is of any order, and f_1, f_2, f_3, &c., rational integral functions of any degrees.

33. Resolve the following determinant into linear factors :–

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_5 & a_1 & a_2 & a_3 & a_4 \\ a_4 & a_5 & a_1 & a_2 & a_3 \\ a_3 & a_4 & a_5 & a_1 & a_2 \\ a_2 & a_3 & a_4 & a_5 & a_1 \end{vmatrix}.$$

In all the rows the constituents are the same five quantities taken in circular order, a different one standing first in each row. A determinant of this kind is called a *circulant*. It is convenient to write a circulant in the form here given, viz., such that the same constituent occupies the diagonal place throughout. Taking θ to be any root of the equation $x^5 - 1 = 0$, and adding to the first column the sum of the constituents of the remaining columns multiplied by $\theta, \theta^2, \theta^3, \theta^4$ respectively, we observe that the following are factors of the determinant :—

$$a_1 + a_2 + a_3 + a_4 + a_5,$$
$$a_1 + \theta a_2 + \theta^2 a_3 + \theta^3 a_4 + \theta^4 a_5,$$
$$a_1 + \theta^2 a_2 + \theta^4 a_3 + \theta a_4 + \theta^3 a_5,$$
$$a_1 + \theta^3 a_2 + \theta a_3 + \theta^4 a_4 + \theta^2 a_5,$$
$$a_1 + \theta^4 a_2 + \theta^3 a_3 + \theta^2 a_4 + \theta a_5,$$

the five roots of $x^5 - 1 = 0$ being $1, \theta, \theta^2, \theta^3, \theta^4$; and comparing the coefficient of a_1^5 in both expressions, it appears that the numerical factor is unity (cf. Ex. 13, Art. 140). A circulant of any order can be treated in a similar manner.

34. The product of two circulants of the same order is a circulant.

35. Calculate the determinant of the n^{th} order

$$\Delta_n \equiv \begin{vmatrix} a_n & b_n & 0 & 0 & 0 & . \\ -1 & a_{n-1} & b_{n-1} & 0 & 0 & . \\ 0 & -1 & a_{n-2} & b_{n-2} & 0 & . \\ 0 & 0 & -1 & a_{n-3} & b_{n-3} & . \\ . & . & . & . & . & . \end{vmatrix},$$

in which all the constituents are zero except those which lie in the diagonal and in lines adjacent to it on either side and parallel to it, one of these latter sets consisting of constituents each equal to -1.

Expanding in terms of the first column, we have the following relation connecting three determinants of the kind here considered whose orders are n, $n-1$, $n-2$:—

$$\Delta_n = a_n \Delta_{n-1} + b_n \Delta_{n-2},$$

By aid of this equation the calculation of any determinant is reduced to that of the two next inferior to it in the series $\Delta_n, \Delta_{n-1}, \Delta_{n-2}, \ldots \Delta_2, \Delta_1$; and the values of Δ_1 and Δ_2 are plainly a_1 and $a_2a_1 + b_2$ respectively.

Dividing the equation just given by Δ_{n-1} we have

$$\frac{\Delta_n}{\Delta_{n-1}} = a_n + \cfrac{b_n}{\cfrac{\Delta_{n-1}}{\Delta_{n-2}}};$$

replacing by a similar value the quotient of Δ_{n-1} by Δ_{n-2}, and continuing the process, it appears that the quotient of any determinant by the one next below it in the series can be expressed as a continued fraction in terms of the given constituents. On account of this property, determinants of the form here treated are called *continuants.* When each of the constituents $b_n, b_{n-1}, \ldots b_3, b_2$ (in the line above the diagonal) is equal to $+1$, the resulting determinant is a *simple continuant.*

36. Calculate the determinant of the n^{th} order

$$\Delta_n \equiv \begin{vmatrix} \alpha & 1 & 0 & 0 & 0 & . \\ \beta & \alpha & 1 & 0 & 0 & . \\ 0 & \beta & \alpha & 1 & 0 & . \\ 0 & 0 & \beta & \alpha & 1 & . \\ . & . & . & . & . & . \end{vmatrix},$$

whose only constituents which do not vanish are $\alpha, \beta, 1$, occupying the diagonal and the lines adjacent and parallel to it as here represented.

The calculation is readily effected for any particular value of n, in a manner similar to that of the last example, by aid of the equation

$$\Delta_n = \alpha\Delta_{n-1} - \beta\Delta_{n-2},$$

the values of Δ_1 and Δ_2 being α and $\alpha^2 - \beta$, respectively.

By examining the formation of the successive values of Δ, the student will readily observe that the terms contained in the result are

$$\alpha^{2r}, \quad \alpha^{2r-2}\beta, \quad \alpha^{2r-4}\beta^2, \ldots \alpha^2\beta^{r-1}, \quad \beta^r,$$

when n is even and of the form $2r$; and

$$\alpha^{2r+1}, \quad \alpha^{2r-1}\beta, \quad \alpha^{2r-3}\beta^2, \ldots \alpha^3\beta^{r-1}, \quad \alpha\beta^r,$$

when n is odd and of the form $2r + 1$.

For the purposes of a subsequent investigation, in which the results just stated will be made use of, it is not necessary to know the general forms of the numerical coefficients which enter into these expressions; but such forms can be arrived at without difficulty, and the following general expression obtained for Δ_n:—

$$\Delta_n = \alpha^n - (n-1)\alpha^{n-2}\beta + \frac{(n-3)(n-2)}{1.2}\alpha^{n-4}\beta^2 - \frac{(n-5)(n-4)(n-3)}{1.2.3}\alpha^{n-6}\beta^3 + \&c.$$

37. When a polynomial U is divided by another U' of lower dimensions, the coefficients of the quotient, and of the remainder, can be expressed as determinants in terms of the coefficients of U and U'.

The method employed in the following particular case is equally applicable in general. Let U be of the fifth, and U' of the third degree; the quotient and remainder can then be represented as follows:—

$$Q = q_0x^2 + q_1x + q_2, \quad R = r_0x^2 + r_1x + r_2.$$

Also, let

$$U = a_0x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5, \quad U' = a'_0x^3 + a'_1x^2 + a'_2x + a'_3.$$

From the identity $$U = QU' + R$$

we have the following equations:—

$$\begin{aligned} a_0 &= q_0a'_0, \\ a_1 &= q_0a'_1 + q_1a'_0, \\ a_2 &= q_0a'_2 + q_1a'_1 + q_2a'_0, \\ a_3 &= q_0a'_3 + q_1a'_2 + q_2a'_1 + r_0, \\ a_4 &= \qquad\quad q_1a'_3 + q_2a'_2 + r_1, \\ a_5 &= \qquad\qquad\qquad q_2a'_3 + r_2. \end{aligned}$$

Solving by Art. 144, q_0, q_1, q_2 are expressed as determinants by means of the first three of these equations; and, taking the first three with each of the others in succession, we determine r_0, r_1, r_2. For example, to find r_0 we have, from the first four equations,

$$\begin{vmatrix} a'_0 & 0 & 0 & -a_0 \\ a'_1 & a'_0 & 0 & -a_1 \\ a'_2 & a'_1 & a'_0 & -a_2 \\ a'_3 & a'_2 & a'_1 & -a_3+r_0 \end{vmatrix} = 0, \text{ or } a'^3_0 r_0 = \begin{vmatrix} a'_0 & 0 & 0 & a_0 \\ a'_1 & a'_0 & 0 & a_1 \\ a'_2 & a'_1 & a'_0 & a_2 \\ a'_3 & a'_2 & a'_1 & a_3 \end{vmatrix}.$$

38. Find the general forms of the coefficients of the quotient, and of the remainder, when a polynomial of even degree $2m$ is divided by a quadratic.

Taking $x^2 + \alpha x + \beta$ as the given quadratic function, we have the identity

$$a_0x^{2m} + a_1x^{2m-1} + a_2x^{2m-2} + \cdots + a_{2m-2}x^2 + a_{2m-1}x + a_{2m}$$
$$\equiv (q_0x^{2m-2} + q_1x^{2m-3} + \cdots + q_{2m-3}x + q_{2m-2})(x^2 + \alpha x + \beta) + r_0x + r_1.$$

Writing down the first $r+1$ equations, formed as in the preceding example, to solve for $q_0, q_1, q_2, \ldots q_r$, it is easily seen that the value of q_r thence derived is a determinant of the r^{th} order of the form treated in Ex. 36, bordered at the top with the constituents $1, 0, \ldots 0, a_0$, and at the right-hand side with $a_0, a_1 \ldots a_r$. Expanding this determinant in terms of the last column, it is immediately seen that any quotient is expressed by means of a series of the determinants of Ex. 36 in the form

$$q_r = a_r - a_{r-1}\Delta_1 + a_{r-2}\Delta_2 - \&\text{c.} \ldots \mp a_1\Delta_{r-1} \pm \Delta_r;$$

the upper or lower sign to be used according as r is even or odd. To obtain the coefficients of the remainder we have the equations

$$\beta q_{2m-3} + \alpha q_{2m-2} + r_0 = a_{2m-1},$$
$$\beta q_{2m-2} + r_1 = a_{2m}.$$

Expressing the values of q_{2m-3}, q_{2m-2} by the formula just proved, and attending to the results of Ex. 36, we derive the following general forms for r_0 and r_1:—

$$r_0 = A_{2m-1} + A_{2m-3}\beta + A_{2m-5}\beta^2 + \ldots + A_3\beta^{m-2} + A_1\beta^{m-1},$$
$$r_1 = a_{2m} \quad + B_{2m-2}\beta + B_{2m-4}\beta^2 + \ldots + B_2\beta^{m-1} + B_0\beta^m,$$

in which the coefficients A, B are all functions of α, the highest power of α in any coefficient A or B being represented by the suffix attached to the coefficient.

39. If the leading constituents of a symmetric determinant be all increased by the same quantity x, the equation in x obtained by equating to zero the determinant so formed has all its roots real.

Let the determinant of the n^{th} order under consideration be denoted by Δ_n, and written in the form

$$\Delta_n = \begin{vmatrix} a+x & h & g & . \\ h & b+x & f & . \\ g & f & c+x & . \\ . & . & . & . \end{vmatrix}.$$

Let the determinant obtained from this by erasing the first row and first column, *i.e.* the leading first minor of Δ_n, be denoted by Δ_{n-1}; again, the leading first minor of Δ_{n-1} by Δ_{n-2}; and so on, the last function Δ_1 obtained in this way being of the form $l + x$. To these we add the positive constant $\Delta_0 = 1$, which may be regarded as completing the series of minors and obtained by the same process, since Δ_n is not altered by affixing a last row and a last column consisting entirely of zero-elements, with the exception of the constituent $+1$ in the leading diagonal. We have now a series of $n+1$ functions—

$$\Delta_n, \ \Delta_{n-1}, \ \Delta_{n-2}, \ \ldots \ \Delta_2, \ \Delta_1, \ \Delta_0,$$

whose degrees in x are represented by the suffixes. When $+\infty$ is substituted for x, the signs are all positive, and when $-\infty$ is substituted, the signs (beginning with Δ_0) are alternately positive and negative. Hence if x be regarded as increasing continuously, n changes of sign must be lost in this series during the passage from $-\infty$ to $+\infty$. Now it appears by the theorem of Art. 149, that a value of x which causes any function (excluding Δ_n, Δ_0) in this series to vanish gives opposite signs to the functions adjacent to it on either side. Δ_0 retains its sign throughout. It follows, exactly as in (2), Art. 96, that a change of sign can never be lost except when x passes through a real root of $\Delta_n = 0$. There must, therefore, exist n real roots of this equation in order that n changes may be lost during the passage of x from $-\infty$ to $+\infty$.

Any equation in the series, being of the same form as $\Delta_n = 0$, has all its roots real. It is plain also that each of these equations is a limiting equation (see Art. 90) with reference to the equation next above it in the series; since, in order that a change of sign may be lost between Δ_n and Δ_{n-1} at the passage through each of two consecutive roots of the former, the value of Δ_{n-1} must change sign between these two values of x. The equation $\Delta_n = 0$ may have equal roots, and by what has been just proved it appears that, when this equation has r roots equal to α, the equation $\Delta_{n-1} = 0$ has $r - 1$ roots equal to α, the equation $\Delta_{n-2} = 0$ has $r - 2$ roots equal to α, and so on.

The determinant here discussed occurs in several investigations in pure and applied mathematics. The proof here given of the important property under discussion is taken from Salmon's *Higher Algebra* (Art. 46), to which work the student is referred for other proofs of the same theorem.

40. If the determinant of the preceding example have r roots equal to α, prove that every first minor has $r - 1$ roots equal to α, every second minor $r - 2$ roots equal to α, and so on.

Employing the notation $A, H, G, \ldots$ for the elements of the reciprocal determinant, we have the equation

$$AB - H^2 = \Delta_{n-2}\,\Delta_n.$$

Now it is easily seen, by proper transpositions of rows and columns, that every *leading* first minor contains the multiple root $r - 1$ times. It follows from the equation just written that the minor H must contain this root $r - 1$ times; and H may be taken to represent any first minor.

41. Find the conditions that the equation

$$\begin{vmatrix} a + x & h & g \\ h & b + x & f \\ g & f & c + x \end{vmatrix} = 0$$

[illegible] have [illegible] roots.

[illegible] minor must contain the double root, we readily derive the required [illegible] in the following form:—

$$a - \frac{gh}{f} = b - \frac{hf}{g} = c - \frac{fg}{h}$$

[illegible] preceding example are taken from Routh's *Dynamics of a System of Rigid Bodies*, Part II., Art. 61.]

42. [illegible] symmetrical determinant can be altered so as to have any selected pair of [illegible] constituents each zero, the determinant remaining symmetrical. [illegible] example, the determinant obtained by putting $x = 0$ in the preceding [illegible] suppose it is required to remove the constituent g. Multiply

each constituent of the third column by a (dividing the whole determinant by a at the same time), and subtract from the constituents so altered those of the first column multiplied by g. Treat now the two corresponding rows in the same way; the resulting determinant is symmetrical, and in it g is replaced by zero. This process may be applied to a determinant of any order, to remove in succession all the conjugate constituents of the first row and column, and afterwards of the remaining rows and columns, so as to reduce the determinant finally to one, all of whose constituents vanish except those in the leading diagonal.

43. Reduce the following determinant, of any order, to a form in which x will appear in the leading constituents only:—

$$\begin{vmatrix} a_1x + a'_1 & b_1x + b'_1 & c_1x + c'_1 & . \\ a_2x + a'_2 & b_2x + b'_2 & c_2x + c'_2 & . \\ a_3x + a'_3 & b_3x + b'_3 & c_3x + c'_3 & . \\ . & . & . & . \end{vmatrix}$$

Multiply by the determinant reciprocal to $(a_1b_2c_3 \ldots l_n)$. If the given determinant is symmetrical, the determinant derived from it in this way will not be symmetrical; but a different process may be used to reduce it in that case to a *symmetrical* determinant which will have x present in the leading constituents only, viz. by removing the coefficients of x from all pairs of conjugate constituents in succession by a process exactly analogous to that of the preceding example. If the coefficients of x in the leading constituents of the reduced determinant should all have the same sign, it may be proved, just as in Ex. 39, that the corresponding equation will have all its roots real.

44. Let a determinant of the n^{th} order be divided into two rectangular arrays, one containing μ rows, and the other ν rows (where $\mu + \nu = n$), and let $\mu\nu$ sums of products be formed by operating with one array on the other as in the multiplication of determinants; if then such relations exist among the constituents that all these sums of products separately vanish, the determinants of order μ formed from the first array are proportional to determinants of order ν formed from the complementary constituents of the second.

To fix the ideas, we take a determinant of the fifth order, but the mode of proof is perfectly general. Let the determinant

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \end{vmatrix}$$

be split horizontally into two arrays, one of three, and the other of two rows; and let the following six relations exist :—

$$\Sigma a_1x_1 = 0,\quad \Sigma a_1y_1 = 0,\quad \Sigma b_1x_1 = 0,\quad \Sigma b_1y_1 = 0,\quad \Sigma c_1x_1 = 0,\quad \Sigma c_1y_1 = 0.$$

If now Δ be expanded by Laplace's theorem, and the minor determinants so taken (as can readily be done) that the expansion is written *with all positive signs*, e. g. in the form :—

$$\Delta \equiv (a_1b_2c_3)(x_4y_5) + (a_1b_3c_4)(x_2y_5) + (a_1b_2c_4)(x_5y_3) + (a_1b_2c_5)(x_3y_4) + \&c.,$$

it is proposed to prove that each minor determinant of the third order formed from the first array is proportional to its factor in the expansion of Δ so written.

We use for convenience the following notation for the expansion last written—

$$\Delta = LL' + MM' + NN' + PP' + \&c.$$

Squaring the determinant Δ, making use of the above relations, replacing by their values the determinants obtained by squaring separately each of the component arrays, and equating the two values of Δ^2 thus obtained, we have

$$(LL' + MM' + NN' + \&c.\ldots)^2 = (L^2 + M^2 + N^2 + \&c.\ldots)(L'^2 + M'^2 + N'^2 + \&c.\ldots),$$

whence

$$(LM' - L'M)^2 + (LN' - L'N)^2 + (MN' - M'N)^2 + \&c.\ldots = 0,$$

from which we have at once

$$\frac{L}{L'} = \frac{M}{M'} = \frac{N}{N'} = \frac{P}{P'} = \&c.$$

45. Write down the relations which exist among the minors of the second order formed from a determinant of the fourth order divided equally into two rectangular arrays in the manner of the last example, like conditions being fulfilled.

We take the general determinant of the fourth order

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix},$$

and first expand it by Laplace's theorem. As the expansion of such a determinant in terms of its second minors is often required in practice, the student is recommended to accustom himself to write it with all positive signs as follows :—

$$(b_1c_2)(a_3d_4) + (c_1a_2)(b_3d_4) + (a_1b_2)(c_3d_4)$$
$$+ (a_1d_2)(b_3c_4) + (b_1d_2)(c_3a_4) + (c_1d_2)(a_3b_4).$$

The method of writing this down is obvious, the same arrangement being observed as on all former occasions where four letters were involved.

By the preceding example, we have at once the relations

$$\frac{(b_1c_2)}{(a_3d_4)} = \frac{(c_1a_2)}{(b_3d_4)} = \frac{(a_1b_2)}{(c_3d_4)} = \frac{(a_1d_2)}{(b_3c_4)} = \frac{(b_1d_2)}{(c_3a_4)} = \frac{(c_1d_2)}{(a_3b_4)},$$

provided the following four equations hold :—

$$\Sigma a_1a_3 = 0, \qquad \Sigma a_1a_4 = 0, \qquad \Sigma a_2a_3 = 0, \qquad \Sigma a_2a_4 = 0.$$

What is here proved has an important application in geometry of three dimensions with reference to the six coordinates of a right line. (See Salmon's *Analytic Geometry of Three Dimensions*, 4th ed., Art. 57 b.)

It may be remarked here that it will be found convenient to write uniformly with positive signs the expansion of a determinant of the third order, which occurs so often in practical questions. Taking, for example, the determinant obtained by erasing the last row and last column of Δ, we write its expansion as follows, the three letters being taken in circular order :—

$$(a_1b_2c_3) \equiv a_1(b_2c_3) + b_1(c_2a_3) + c_1(a_2b_3).$$

46. Derive the equations (3) of Art. 145, for obtaining the ratios of n variables from $n - 1$ linear homogeneous equations, from the proposition of Ex. 44.

47. Express by determinants the values of the unknown quantities derived from a set of given linear equations by the *Method of Least Squares*.

The given equations, which are greater in number than the unknown quantities, are supposed to have been obtained as the result of observation or experiment; and the numerical coefficients which enter into them, being consequently liable to errors of observation, are not known with certainty. In such cases the most reliable values of the unknown quantities are obtained in the manner about to be explained by what is called the "method of least squares." Take, for example, five equations of the form $a_1x + b_1y + c_1z = m_1$, $a_2x + b_2y + c_2z = m_2$, &c., between three unknown quantities x, y, z. Multiply them respectively by a_1, a_2, a_3, a_4, a_5, and add; again by b_1, b_2, b_3, b_4, b_5, and add; and again by c_1, c_2, c_3, c_4, c_5, and add. In this way the following three equations are obtained :—

$$x\Sigma a_1^2 + y\Sigma a_1b_1 + z\Sigma a_1c_1 = \Sigma a_1m_1,$$

$$x\Sigma a_1b_1 + y\Sigma b_1^2 + z\Sigma b_1c_1 = \Sigma b_1m_1,$$

$$x\Sigma a_1c_1 + y\Sigma b_1c_1 + z\Sigma c_1^2 = \Sigma c_1m_1;$$

from which we have, without difficulty,

$$x = \frac{(a_1b_2c_3)(m_1b_2c_3) + (a_1b_2c_4)(m_1b_2c_4) + \ldots + (a_3b_4c_5)(m_3b_4c_5)}{(a_1b_2c_3)^2 + (a_1b_2c_4)^2 + \ldots + (a_3b_4c_5)^2},$$

with corresponding values for y and z, each of these values containing ten terms in the numerator and ten in the denominator.

48. Show that the value of x given in the preceding example can be obtained by first eliminating y and z from every set of three of the five given equations, and then applying the method of least squares to the ten equations in x alone which result from the elimination.

CHAPTER XIV.

ELIMINATION.

150. **Definitions.**—Being given a system of n equations, homogeneous between n variables, or non-homogeneous between $n-1$ variables, if we combine these equations in such a manner as to eliminate the variables, and obtain an equation $R=0$ containing only the coefficients of the equations, the quantity R is, when expressed in a rational and integral form, called the *Resultant* or *Eliminant*.

In what follows we shall be concerned chiefly with two equations involving one unknown quantity x only. In this case the equation $R=0$ asserts that the two equations are consistent; that is, they are both satisfied by a common value of x. We now proceed to show how the elimination may be performed so as to obtain the quantity R, illustrating the different methods by simple examples. It is proper to observe that the value of R arrived at by some processes of elimination may contain a redundant factor. The method of elimination by symmetric functions leads to a value of R free from any such factor; and we refer, therefore, to the conclusion of the discussion in the next Article for the precise definition of the *Resultant*.

Let it be required to eliminate x between the equations

$$ax^2+2bx+c=0, \quad a'x^2+2b'x+c'=0.$$

Solving these equations, and equating the values of x so obtained, the result of elimination appears in the irrational form

$$-\frac{b}{a}+\frac{\sqrt{b^2-ac}}{a}=-\frac{b'}{a'}+\frac{\sqrt{b'^2-a'c'}}{a'}.$$

Multiplying by aa' we obtain

$$ab' - a'b = a\sqrt{b'^2 - a'c'} - a'\sqrt{b^2 - ac}.$$

Squaring both sides, and dividing by the redundant factor aa', and then squaring again, we find

$$R = 4(ac - b^2)(a'c' - b'^2) - (ac' + a'c - 2bb')^2.$$

This method of forming the resultant is very limited in application, as it is not, in general, possible to express by an algebraic formula a root of an equation higher than the fourth degree. Other methods have consequently been devised for determining the resultant without first solving the equations. We now proceed to explain the method of elimination by symmetric functions of the roots of the equations.

151. **Elimination by Symmetric Functions.**—Let two algebraic equations of the m^{th} and n^{th} degrees be

$$\phi(x) \equiv a_0x^m + a_1x^{m-1} + a_2x^{m-2} + \ldots + a_m = 0,$$
$$\psi(x) \equiv b_0x^n + b_1x^{n-1} + b_2x^{n-2} + \ldots + b_n = 0;$$

and let it be required to find the condition that these equations should have a common root. For this purpose let the roots of the equation $\phi(x) = 0$ be $\alpha_1, \alpha_2, \ldots \alpha_m$. If the given equations have a common root, it is *necessary* and *sufficient* that one of the quantities

$$\psi(\alpha_1), \quad \psi(\alpha_2), \quad \ldots, \quad \psi(\alpha_m)$$

should be zero, or, in other words, that the product

$$\psi(\alpha_1)\psi(\alpha_2)\psi(\alpha_3) \ldots \psi(\alpha_m)$$

should vanish. If, now, we transform this product into a rational and integral function of the coefficients, which is always possible, as it is a symmetric function of the roots of the equation $\phi(x) = 0$, we shall have the resultant required. Further, if $\beta_1, \beta_2, \ldots \beta_n$ be the roots of the equation $\psi(x) = 0$, we have

$$\psi(\alpha_1) = b_0(\alpha_1 - \beta_1)(\alpha_1 - \beta_2) \ldots (\alpha_1 - \beta_n),$$
$$\psi(\alpha_2) = b_0(\alpha_2 - \beta_1)(\alpha_2 - \beta_2) \ldots (\alpha_2 - \beta_n),$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\psi(\alpha_m) = b_0(\alpha_m - \beta_1)(\alpha_m - \beta_2) \ldots (\alpha_m - \beta_n).$$

If we change the signs of the mn factors, and multiply these equations, taking together the factors which are situated in the same column, we find

$$a_0^n\psi(\alpha_1)\psi(\alpha_2)\ldots\psi(\alpha_m) = (-1)^{mn}b_0^m\phi(\beta_1)\phi(\beta_2)\ldots\phi(\beta_n).$$

We may therefore take

$$R = (-1)^{mn}b_0^m\phi(\beta_1)\phi(\beta_2)\ldots\phi(\beta_n) = a_0^n\psi(\alpha_1)\psi(\alpha_2)\ldots\psi(\alpha_m), \quad (1)$$

for both these values of R are integral functions of the coefficients of $\phi(x)$ and $\psi(x)$, which vanish only when $\phi(x)$ and $\psi(x)$ have a common factor, and which become identical when they are expressed in terms of the coefficients.

152. **Properties of the Resultant.**—(1). *The order of the resultant of two equations in the coefficients is equal to the sum of the degrees of the equations, the coefficients of the first equation entering R in the degree of the second, and the coefficients of the second entering in the degree of the first.*

This appears by reviewing the two forms of R in (1), Art. 151; for in the first form $a_0, a_1, \ldots a_m$ enter in the n^{th} degree, and in the second form $b_0, b_1 \ldots b_n$ enter in the m^{th} degree. Also, it may be seen that two terms, one selected from each form, are $(-1)^{mn}b_0^m a_m^n$ and $a_0^n b_n^m$.

(2). *If the roots of both equations be multiplied by the same quantity ρ, the resultant is multiplied by ρ^{mn}.*

This is evident, since any one of the mn factors of the form $\alpha_p - \beta_q$ becomes $\rho(\alpha_p - \beta_q)$, and therefore ρ^{mn} divides the resultant. From this we may conclude that *the weight of the resultant is mn*, in which form this proposition is often stated.

(3). *If the roots of both equations be increased by the same quantity, the resultant of the equations so transformed is equal to the resultant of the original equations.*

For we have

$$\pm R = a_0^n b_0^m \Pi(\alpha_p - \beta_q),$$

where Π signifies the continued product of the mn terms of the form $\alpha_p - \beta_q$; and this is unaltered when α_p and β_q receive the same increment.

(4). *If the roots be changed into their reciprocals, the value of R obtained from the transformed equations remains unaltered, except in sign when mn is an odd number.*

Making this transformation in

$$R = a_0^n b_0^m \Pi (\alpha_p - \beta_q),$$

we have

$$R' = a_m^n b_n^m (-1)^{mn} \frac{\Pi (\alpha_p - \beta_q)}{(\alpha_1 \alpha_2 \ldots \alpha_m)^n (\beta_1 \beta_2 \ldots \beta_n)^m};$$

but

$$\alpha_1 \alpha_2 \ldots \alpha_m = (-1)^m \frac{a_m}{a_0}, \quad \beta_1 \beta_2 \ldots \beta_n = (-1)^n \frac{b_n}{b_0};$$

substituting, we obtain

$$R' = a_0^n b_0^m (-1)^{mn} \Pi (\alpha_p - \beta_q) = (-1)^{mn} R.$$

From this it follows that in the resultant of two equations the coefficients with complementary suffixes of both equations, e.g. a_0, a_m; a_1, a_{m-1}, &c., may be all interchanged without altering the value of the resultant.

(5). *If both equations be transformed by homographic transformation, that is, by substituting for x*

$$\frac{\lambda x + \mu}{\lambda' x + \mu'},$$

and each simple factor multiplied by $\lambda' x + \mu'$, *to render the new equations integral, then the new resultant* $R' = (\lambda \mu' - \lambda' \mu)^{mn} R$.

To prove this, we have

$$\phi (x) = a_0 (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_m),$$
$$\psi (x) = b_0 (x - \beta_1)(x - \beta_2) \ldots (x - \beta_n);$$

also

$$x - \alpha_r \text{ becomes } (\lambda - \lambda' \alpha_r)\left(x - \frac{\mu' \alpha_r - \mu}{\lambda - \lambda' \alpha_r}\right),$$

$$x - \beta_r \quad ,, \quad (\lambda - \lambda' \beta_r)\left(x - \frac{\mu' \beta_r - \mu}{\lambda - \lambda' \beta_r}\right).$$

Multiplying together all the factors of each equation,

$$a_0 \text{ becomes } a_0 (\lambda - \lambda' \alpha_1)(\lambda - \lambda' \alpha_2) \ldots (\lambda - \lambda' \alpha_m),$$
$$b_0 \quad ,, \quad b_0 (\lambda - \lambda' \beta_1)(\lambda - \lambda' \beta_2) \ldots (\lambda - \lambda' \beta_n).$$

Also, since a_r, β_r are transformed into $\frac{\mu' a_r - \mu}{\lambda - \lambda' a_r}$, $\frac{\mu'\beta_r - \mu}{\lambda - \lambda'\beta_r}$,

$$a_r - \beta_r \text{ becomes } \frac{(\lambda\mu' - \lambda'\mu)(a_r - \beta_r)}{(\lambda - \lambda' a_r)(\lambda - \lambda'\beta_r)};$$

whence

$$a_0^n b_0^m \Pi (a_r - \beta_r) \text{ becomes } a_0^n b_0^m (\lambda\mu' - \lambda'\mu)^{mn} \Pi (a_r - \beta_r),$$

that is, the resultant calculated from the new forms of $\phi(x)$ and $\psi(x)$ is

$$(\lambda\mu' - \lambda'\mu)^{mn} R.$$

This proposition includes the three foregoing; and they are collectively equivalent to the present proposition.

153. **Euler's Method of Elimination.**—When two equations $\phi(x) = 0$, and $\psi(x) = 0$, of the m^{th} and n^{th} degrees respectively, have any common root θ, we may assume

$$\phi(x) \equiv (x - \theta)\, \phi_1(x),$$

$$\psi(x) \equiv (x - \theta)\, \psi_1(x),$$

where

$$\phi_1(x) \equiv p_1 x^{m-1} + p_2 x^{m-2} + \ldots + p_m,$$

$$\psi_1(x) \equiv q_1 x^{n-1} + q_2 x^{n-2} + \ldots + q_n,$$

the coefficients being undetermined quantities depending on θ. Whence we have

$$\phi(x)\, \psi_1(x) \equiv \psi(x)\, \phi_1(x),$$

an identical equation of the $(m+n-1)^{th}$ degree. Now, equating the coefficients of the different powers of x on both sides of the equation, we have $m + n$ homogeneous equations of the first degree in the $m + n$ quantities $p_1, p_2, \ldots p_m, q_1, q_2, \ldots q_n$; and eliminating these quantities by the method of Art. 145, we obtain the resultant of the two given equations in the form of a determinant.

EXAMPLE.

Suppose the two equations

$$ax^2 + bx + c = 0, \quad a_1x^2 + b_1x + c_1 = 0$$

to have a common root. We have identically

$$(q_1x + q_2)(ax^2 + bx + c) \equiv (p_1x + p_2)(a_1x^2 + b_1x + c_1),$$

or

$$(q_1a - p_1a_1)x^3 + (q_1b + q_2a - p_1b_1 - p_2a_1)x^2 + (q_1c + q_2b - p_1c_1 - p_2b_1)x + q_2c - p_2c_1 \equiv 0.$$

Equating to zero all the coefficients of this equation, we have the four homogeneous equations

$$\begin{aligned} q_1a \qquad\quad - p_1a_1 \qquad\quad &= 0, \\ q_1b + q_2a - p_1b_1 - p_2a_1 &= 0, \\ q_1c + q_2b - p_1c_1 - p_2b_1 &= 0, \\ q_2c \qquad\quad - p_2c_1 &= 0; \end{aligned}$$

and eliminating p_1, p_2 q_1, q_2, we obtain the condition for a common root in the form

$$\begin{vmatrix} a & 0 & a_1 & 0 \\ b & a & b_1 & a_1 \\ c & b & c_1 & b_1 \\ 0 & c & 0 & c_1 \end{vmatrix} = 0.$$

The student can easily verify that this result is the same as that of Art. 150.

154. Sylvester's Dialytic Method of Elimination.— This method leads to the same determinants for resultants as the method of Euler just explained; but it has an advantage over Euler's method in point of generality, since it can often be applied to form the resultant of equations involving several variables.

Suppose we require the resultant of the two equations

$$\phi(x) \equiv a_0x^m + a_1x^{m-1} + a_2x^{m-2} + \ldots + a_m = 0,$$

$$\psi(x) \equiv b_0x^n + b_1x^{n-1} + b_2x^{n-2} + \ldots + b_n = 0,$$

we multiply the first by the successive powers of x,

$$x^{n-1}, \ x^{n-2}, \ \ldots x^2, \ x, \ x^0;$$

and the second by $x^{m-1}, x^{m-2}, \ldots x^2, x, x^0,$

thus obtaining $m + n$ equations, the highest power of x being $m + n - 1$. We have, consequently, equations enough from which to eliminate x^{m+n-1}, x^{m+n-2}, $\ldots x^2$, x considered as distinct variables.

EXAMPLES.

1. Find the resultant R of two quadratic equations

$$ax^2 + bx + c = 0, \quad a_1x^2 + b_1x + c_1 = 0.$$

We have

$$\begin{aligned} ax^3 + bx^2 + cx \quad\quad &= 0, \\ ax^2 + bx + c &= 0, \\ a_1x^3 + b_1x^2 + c_1x \quad\quad &= 0, \\ a_1x^2 + b_1x + c_1 &= 0; \end{aligned}$$

from which, eliminating x^3, x^2, x, we get the same determinant as in the preceding Article, columns now replacing rows:

$$R \equiv \begin{vmatrix} a & b & c & 0 \\ 0 & a & b & c \\ a_1 & b_1 & c_1 & 0 \\ 0 & a_1 & b_1 & c_1 \end{vmatrix}$$

2. Form the resultant of the two equations

$$U \equiv a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 = 0,$$
$$V \equiv b_0 + b_1x + b_2x^2 + b_3x^3 = 0.$$

Proceeding as above, we easily find

$$R \equiv \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & 0 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 \\ b_0 & b_1 & b_2 & b_3 & 0 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & b_3 & 0 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 & 0 \\ 0 & 0 & 0 & b_0 & b_1 & b_2 & b_3 \end{vmatrix}.$$

It will be observed that R contains the coefficients of U in the 3rd degree, and those of V in the 4th degree; also $a_0{}^3b_3{}^4$ is a term in R (see (1), Art. 152).

155. Bezout's Method of Elimination.—The general method will be most easily understood by applying it in the first instance to particular cases. We proceed to this application—(1) when the equations are of the same degree, and (2) when they are of different degrees.

(1). Let us take the two cubic equations

$$ax^3 + bx^2 + cx + d = 0, \quad a_1x^3 + b_1x^2 + c_1x + d_1 = 0.$$

Multiplying these two equations successively by

$$a_1 \quad \text{and} \quad a,$$
$$a_1x + b_1 \quad ,, \quad ax + b,$$
$$a_1x^2 + b_1x + c_1 \quad ,, \quad ax^2 + bx + c,$$

and subtracting each time the products so formed, we find the three following equations :—

$$(ab_1)x^2 \qquad + (ac_1)\,x + (ad_1) = 0,$$
$$(ac_1)x^2 + \{(ad_1) + (bc_1)\}x + (bd_1) = 0,$$
$$(ad_1)x^2 \qquad + (bd_1)\,x + (cd_1) = 0.$$

By eliminating from these equations x^2, x, as distinct variables, the resultant is obtained in the form of a symmetrical determinant as follows :—

$$\begin{vmatrix} (ab_1) & (ac_1) & (ad_1) \\ (ac_1) & (ad_1) + (bc_1) & (bd_1) \\ (ad_1) & (bd_1) & (cd_1) \end{vmatrix}$$

To render the law of formation of the resultant more apparent, the following mode of procedure is given :—

Let the two equations be biquadratics, as follows :—

$$ax^4 + bx^3 + cx^2 + dx + e = 0, \quad a_1x^4 + b_1x^3 + c_1x^2 + d_1x + e_1 = 0\,;$$

whence, following Cauchy's mode of presenting Bezout's method, we have the system of equations

$$\frac{a}{a_1} = \frac{bx^3 + cx^2 + dx + e}{b_1x^3 + c_1x^2 + d_1x + e_1},$$
$$\frac{ax + b}{a_1x + b_1} = \frac{cx^2 + dx + e}{c_1x^2 + d_1x + e_1},$$
$$\frac{ax^2 + bx + c}{a_1x^2 + b_1x + c_1} = \frac{dx + e}{d_1x + e_1},$$
$$\frac{ax^3 + bx^2 + cx + d}{a_1x^3 + b_1x^2 + c_1x + d_1} = \frac{e}{e_1},$$

which, when rendered integral, lead, on the elimination of x^3, x^2, x, to the following form for the resultant:—

$$\begin{vmatrix} (ab_1) & (ac_1) & (ad_1) & (ae_1) \\ (ac_1) & (ad_1) + (bc_1) & (ae_1) + (bd_1) & (be_1) \\ (ad_1) & (ae_1) + (bd_1) & (be_1) + (cd_1) & (ce_1) \\ (ae_1) & (be_1) & (ce_1) & (de_1) \end{vmatrix}.$$

If, now, we consider the two symmetrical determinants

$$\begin{vmatrix} (ab_1) & (ac_1) & (ad_1) & (ae_1) \\ (ac_1) & (ad_1) & (ae_1) & (be_1) \\ (ad_1) & (ae_1) & (be_1) & (ce_1) \\ (ae_1) & (be_1) & (ce_1) & (de_1) \end{vmatrix}, \quad \begin{vmatrix} (bc_1) & (bd_1) \\ (bd_1) & (cd_1) \end{vmatrix},$$

the formation of which is at once apparent, we observe that R is obtained by adding the constituents of the second to the four central constituents of the first.

Similarly, in the case of the two equations of the fifth degree

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0,$$

$$a_1x^5 + b_1x^4 + c_1x^3 + d_1x^2 + e_1x + f_1 = 0,$$

the resultant is obtained from the three following determinants:—

$$\begin{vmatrix} (ab_1) & (ac_1) & (ad_1) & (ae_1) & (af_1) \\ (ac_1) & (ad_1) & (ae_1) & (af_1) & (bf_1) \\ (ad_1) & (ae_1) & (af_1) & (bf_1) & (cf_1) \\ (ae_1) & (af_1) & (bf_1) & (cf_1) & (df_1) \\ (af_1) & (bf_1) & (cf_1) & (df_1) & (ef_1) \end{vmatrix}, \quad \begin{vmatrix} (bc_1) & (bd_1) & (be_1) \\ (bd_1) & (be_1) & (ce_1) \\ (be_1) & (ce_1) & (de_1) \end{vmatrix}, \quad (cd_1),$$

by adding the constituents of the second to the nine central constituents of the first, and then adding the third to the central constituent of the determinant so formed. The student will have no difficulty in applying a similar process of superposition to the formation of the determinant in general.

(2). We take now the case of two equations of different dimensions, for example,

$$ax^4 + bx^3 + cx^2 + dx + e = 0, \quad a_1x^2 + b_1x + c_1 = 0.$$

Multiplying these equations successively by

$$a_1 \quad \text{and} \quad ax^2,$$
$$a_1x + b_1 \quad ,, \quad (ax + b)x^2,$$

and subtracting each time the products so formed, we find the two following equations :—

$$(ab_1)\,x^3 + (ac_1)\,x^2 - da_1x - ea_1 = 0,$$
$$(ac_1)\,x^3 + \{(bc_1) - da_1\}\,x^2 - \{db_1 + ea_1\}\,x - eb_1 = 0.$$

If, now, we join to these the two equations

$$a_1x^3 + b_1x^2 + c_1x = 0,$$
$$a_1x^2 + b_1x + c_1 = 0,$$

we shall have four equations by means of which x^3, x^2, x can be eliminated; whence we obtain the resultant in the form of a determinant as follows :—

$$\begin{vmatrix} (ab_1) & (ac_1) & da_1 & ea_1 \\ (ac_1) & (bc_1) - da_1 & db_1 + ea_1 & eb_1 \\ a_1 & b_1 & -c_1 & 0 \\ 0 & a_1 & -b_1 & -c_1 \end{vmatrix}.$$

This determinant involves the coefficients of the first equation in the second degree, and the coefficients of the second equation in the fourth degree, as it should do; whence no extraneous factor enters this form of the resultant.

We now proceed to the general case of two equations of the m^{th} and n^{th} degrees.

Let the equations be

$$\phi(x) \equiv a_0x^m + a_1x^{m-1} + a_2x^{m-2} + \ldots + a_m = 0,$$
$$\psi(x) \equiv b_0x^n + b_1x^{n-1} + b_2x^{n-2} + \ldots + b_n = 0,$$

where $m > n$; and let the second equation be multiplied by x^{m-n}. We have then

$$b_0x^m + b_1x^{m-1} + b_2x^{m-2} + \dots + b_nx^{m-n} = 0,$$

an equation of the same degree as the first. This equation has, however, in addition to the n roots of $\psi(x) = 0$, $m - n$ zero roots; so that we must be on our guard lest the factor $a_m{}^{m-n}$ (i.e. the result of substituting these roots in $\phi(x)$) enter the form of the resultant obtained. From these two equations we derive, as in the above case—(1), the following n equations:—

$$\frac{a_0}{b_0} = \frac{a_1x^{m-1} + a_2x^{m-2} + \dots + a_m}{b_1x^{m-1} + b_2x^{m-2} + \dots + b_nx^{m-n}},$$

$$\frac{a_0x + a_1}{b_0x + b_1} = \frac{a_2x^{m-2} + a_3x^{m-3} + \dots + a_m}{b_2x^{m-2} + b_3x^{m-3} + \dots + b_nx^{m-n}},$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots;$$

$$\frac{a_0x^{n-1} + a_1x^{n-2} + \dots + a_{n-1}}{b_0x^{n-1} + b_1x^{n-2} + \dots + b_{n-1}} = \frac{a_nx^{m-n} + a_{n+1}x^{m-n-1} + \dots + a_m}{b_nx^{m-n}},$$

which, when rendered integral, are all of the $(m-1)^{th}$ degree; whence, eliminating $x^{m-1}, x^{m-2}, \dots x$ as independent quantities between these n and the $m - n$ equations

$$\begin{aligned} b_0x^{m-1} + b_1x^{m-2} + b_2x^{m-3} + \dots \quad &= 0, \\ b_0x^{m-2} + b_1x^{m-3} + \dots \quad &= 0, \\ \dots \quad \dots \quad \dots \quad & \\ b_0x^n + b_1x^{n-1} + \dots + b_n &= 0, \end{aligned}$$

we obtain the resultant in the form of a determinant of the m^{th} order, the coefficients of the first equation entering in the degree n, and the coefficients of the second equation entering in the degree m; whence it appears that no extraneous factor can enter; and that the resultant as obtained by this method has not been affected by the introduction of the zero roots.

If R be the resultant of two equations $\phi(x) = 0$, $\psi(x) = 0$, whose degrees are both equal to m, the resultant R' of the system

$$\lambda\phi(x) + \mu\psi(x) = 0, \quad \lambda'\phi(x) + \mu'\psi(x) = 0$$

is
$$(\lambda\mu' - \lambda'\mu)^m R;$$

for each of the minors $(a_r b_s)$, which in Bezout's method constitute the determinant form of R, becomes in this case

$$\begin{vmatrix} \lambda a_r + \mu b_r, & \lambda' a_r + \mu' b_r \\ \lambda a_s + \mu b_s, & \lambda' a_s + \mu' b_s \end{vmatrix} = (\lambda\mu' - \lambda'\mu)(a_r b_s);$$

whence $R' = (\lambda\mu' - \lambda'\mu)^m R$, since R is a determinant of order m.

156. **Other Methods of Elimination.**—We conclude the subject of Elimination with an account of a method which is often employed, but which has the disadvantage of giving the resultant multiplied in general by extraneous factors. The process about to be explained is virtually equivalent to that usually described as the method of the greatest common measure.

In forming by this method the resultant of the two quadratic equations

$$ax^2 + bx + c = 0, \quad a_1x^2 + b_1x + c_1 = 0,$$

we multiply these equations successively by a_1 and a, c_1 and c, and subtract the products so formed. We thus find the two equations

$$(ab_1)x + (ac_1) = 0,$$
$$x\{(ac_1)x + (bc_1)\} = 0.$$

Observing that the value zero for x does not satisfy both the given equations, we may discard the factor x from the second of the equations last written, and thus obtain the resultant without any extraneous factor in the form

$$(ac_1)^2 - (ab_1)(bc_1) = 0.$$

As the degree of this expression is four, and its weight four, it is a correct form for the resultant.

To form by the same process the resultant of the cubic equations

$$ax^3 + bx^2 + cx + d = 0, \quad a_1x^3 + b_1x^2 + c_1x + d_1 = 0,$$

we multiply these equations successively by a_1 and a, d_1 and d, and subtract each time the products so formed. We have then

$$(ab_1)x^2 + (ac_1)x + (ad_1) = 0, \quad (ad_1)x^2 + (bd_1)x + (cd_1) = 0. \quad (1)$$

Now, eliminating x between these two quadratics by means of the formula above obtained, we find for their resultant

$$\begin{vmatrix} (ab_1) & (ad_1) \\ (ad_1) & (cd_1) \end{vmatrix}^2 - \begin{vmatrix} (ab_1) & (ac_1) \\ (ad_1) & (bd_1) \end{vmatrix} \times \begin{vmatrix} (ac_1) & (ad_1) \\ (bd_1) & (cd_1) \end{vmatrix},$$

an expression whose degree is 8 and weight 12, in place of degree 6 and weight 9; whence it appears that it ought to be divisible by a factor whose degree is 2 and weight 3. This factor must therefore be of the form $l(bc_1) + m(ad_1)$. We proceed now to show that it is (ad_1); and to find the quotient when this factor is removed.

For this purpose, retaining only the terms which do not directly involve (ad_1), we have

$$(ab_1)(cd_1)\{(ab_1)(cd_1) + (ca_1)(bd_1)\},$$

which is divisible by (ad_1), since

$$(bc_1)(ad_1) + (ca_1)(bd_1) + (ab_1)(cd_1) = 0.$$

Expanding the determinants, and dividing off by (ad_1), we find ultimately the quotient

$$(ad_1)^3 - 2(ab_1)(cd_1)(ad_1) + (bd_1)(ca_1)(ad_1)$$
$$+ (ca_1)^2(cd_1) + (ab_1)(bd_1)^2 - (ab_1)(bc_1)(dc_1),$$

which, being of the proper degree and weight, is the resultant.

If we proceed in a similar manner to form the resultant of two biquadratic equations, by reducing the process to an elimination between two cubic equations, we shall have to remove an extraneous factor of the fourth degree, which is the condition that these cubics should have a common factor when the biquadratics from which they are derived have not necessarily a common factor; and in general, if we seek by this method the resultant of two equations of the n^{th} degree, eliminating between two equations of the $(n-1)^{th}$ degree, we shall have to remove an extraneous factor of the order $2n - 4$. This method, therefore, is inferior to all the preceding methods; and it cannot be conveniently used except when, from the nature of the investigation, extraneous factors can be easily removed.

157. **Discriminants.**—The *discriminant* of an equation involving a single unknown is the simplest function of the coefficients, in a rational and integral form, whose vanishing expresses the condition for equal roots. We have had examples of such functions in Arts. 43 and 68. We proceed to show that they come under eliminants as particular cases.

If an equation $f(x) = 0$ has a double root, this root must occur once in the equation $f'(x) = 0$; and subtracting $xf'(x)$ from $nf(x)$, the same root must occur in the equation $nf(x) - xf'(x) = 0$. This is an equation of the $(n-1)^{th}$ degree in x; and by eliminating x between it and the equation $f'(x) = 0$, which is also of the $(n-1)^{th}$ degree, we obtain a function of the coefficients whose vanishing expresses the condition for equal roots. The degree of this eliminant in the coefficients of $f(x)$ is $2(n-1)$; and its weight is $n(n-1)$, as may be easily seen by examining the specimen terms given in section (1), Art. 152. Expressed as a symmetric function of the roots of the given equation, the discriminant will be the product of all the differences in the lowest power which can be expressed in a rational form in terms of the coefficients. Now the product of the squares of the differences $\Pi(\alpha_1 - \alpha_2)^2$ can be so expressed; and since it is of the $2(n-1)^{th}$ degree in any one root, and of the $n(n-1)^{th}$ degree in all the roots, it follows that the discriminant multiplied by a numerical factor is equal to $a_0^{2(n-1)}\,\Pi(\alpha_1 - \alpha_2)^2$.

If the function $f(x)$ be made homogeneous by the introduction of a second variable y, the two functions whose resultant is the discriminant of $f(x)$ are the differential coefficients of $f(x)$ with regard to x and y respectively. In the same way, in general, the discriminant of a function homogeneous in any number n of variables is the result of eliminating the variables from the n equations obtained by differentiating with regard to each variable in turn.

EXAMPLES.

1. Find the discriminant of

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0.$$

We have here to find the eliminant of the two equations

$$a_0x^2 + 2a_1x + a_2 = 0,$$
$$a_1x^2 + 2a_2x + a_3 = 0.$$

The condition for a common root is, by Art. 150,

$$4(a_0a_2 - a_1^2)(a_1a_3 - a_2^2) - (a_0a_3 - a_1a_2)^2 = 0.$$

The function of the coefficients here obtained is therefore the discriminant, which may also be written in the form of a determinant, as follows, by Art. 154:

$$\begin{vmatrix} a_0 & 2a_1 & a_2 & 0 \\ 0 & a_0 & 2a_1 & a_2 \\ a_1 & 2a_2 & a_3 & 0 \\ 0 & a_1 & 2a_2 & a_3 \end{vmatrix}.$$

It can be easily verified that this value of the discriminant is the same as that already obtained in Art. 42.

2. Express as a determinant the discriminant of the biquadratic

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0.$$

We have here to eliminate x from the equations

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0,$$
$$a_1x^3 + 3a_2x^2 + 3a_3x + a_4 = 0.$$

By the method of Art. 154 the resultant is

$$\begin{vmatrix} a_0 & 3a_1 & 3a_2 & a_3 & 0 & 0 \\ 0 & a_0 & 3a_1 & 3a_2 & a_3 & 0 \\ 0 & 0 & a_0 & 3a_1 & 3a_2 & a_3 \\ a_1 & 3a_2 & 3a_3 & a_4 & 0 & 0 \\ 0 & a_1 & 3a_2 & 3a_3 & a_4 & 0 \\ 0 & 0 & a_1 & 3a_2 & 3a_3 & a_4 \end{vmatrix}.$$

This must be the same as $I^3 - 27J^2$ of Art. 68.

3. Express the discriminant of the quartic as a determinant by Bezout's method of elimination.

4. Prove that the discriminant, Δ_m, of the equation

$$U \equiv ax^m + by^m + cz^m = 0,$$

where

$$x + y + z = 0,$$

may be obtained by rendering rational, in the form $\Delta_m = 0$, the equation

$$(bc)^{\frac{1}{m-1}} + (ca)^{\frac{1}{m-1}} + (ab)^{\frac{1}{m-1}} = 0;$$

and calculate in particular the values of Δ_3, Δ_4, Δ_5.

When z is replaced by its value from $x + y + z = 0$, the given function U contains two variables, and the discriminant is obtained by eliminating x and y from

$$\frac{\partial U}{\partial x} = 0 \quad \text{and} \quad \frac{\partial U}{\partial y} = 0.$$

5. Prove by elimination that $J = 0$ is one condition for the equality of three roots of the biquadratic of Ex. 2.

Since the triple root must be a double root of

$$U_3 \equiv a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0,$$

and therefore a single root of $a_1x^2 + 2a_2x + a_3 = 0$; and since it must also be a single root of

$$U_2 \equiv a_0x^2 + 2a_1x + a_2 = 0,$$

it follows from the identity

$$U_4 \equiv x^2U_2 + 2x(a_1x^2 + 2a_2x + a_3) + a_2x^2 + 2a_3x + a_4$$

that the triple root must be a root common to the three equations

$$a_0x^2 + 2a_1x + a_2 = 0,$$

$$a_1x^2 + 2a_2x + a_3 = 0,$$

$$a_2x^2 + 2a_3x + a_4 = 0.$$

Hence the condition

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} \equiv J = 0.$$

6. Prove that the discriminant of the product of two functions is the product of their discriminants multiplied by the square of their eliminant.

This appears by applying the results of Art. 151 and the present Article; for the product of the squares of the differences of all the roots is made up of the product of the squares of the differences of the roots of each equation separately and the square of the product of the differences formed by taking each root of one equation with all the roots of the other.

158. **Determination of a Root common to two Equations.**—If R be the resultant of two equations

$$U = a_m x^m + a_{m-1} x^{m-1} + \ldots + a_0 = 0,$$
$$V = b_n x^n + b_{n-1} x^{n-1} + \ldots + b_0 = 0,$$

and a any common root, then

$$a = \frac{\dfrac{dR}{da_1}}{\dfrac{dR}{da_0}} = \frac{\dfrac{dR}{da_2}}{\dfrac{dR}{da_1}} = \frac{\dfrac{dR}{da_3}}{\dfrac{dR}{da_2}} = \&c.$$

To prove this, we first show that functions $\phi(x)$ and $\psi(x)$ can be obtained such that $R = U\phi(x) + V\psi(x)$, namely, when U and V are multiplied by $\phi(x)$ and $\psi(x)$, respectively, and added, all terms involving x vanish identically. Take, for example, the form of R given for two functions of the 4th and 3rd degrees, respectively, in Ex. 2, Art. 154. Multiply the second column by x, the third by x^2, &c., and add to the first column, thus obtaining $U, xU, x^2U, V, xV, x^2V, x^3V$ for the constituents of the first column. The determinant when expanded takes then the form $U\phi(x) + V\psi(x)$, where ϕ is a quadratic function, and ψ a cubic function of x. This mode of proof can be applied to any two functions; and it will be observed in the general case that ϕ and ψ are of the degrees $n - 1$ and $m - 1$, respectively, the degrees of U and V being m and n. We have therefore

$$R = U\phi + V\psi;$$

whence
$$\frac{dR}{da_p} = x^p\phi + U\frac{d\phi}{da_p} + V\frac{d\psi}{da_p},$$
$$\frac{dR}{da_{p+1}} = x^{p+1}\phi + U\frac{d\phi}{da_{p+1}} + V\frac{d\psi}{da_{p+1}};$$

and when a is a common root of the equations $U = 0$, and $V = 0$, we have, substituting this value for x in the preceding equations,

$$a\frac{dR}{da_p} = \frac{dR}{da_{p+1}},$$

which proves the proposition.

A double root of an equation can be determined in a similar manner by differentiating the discriminant Δ.

When the equations $U=0$ and $V=0$ have two roots common, the first differential coefficients of R with regard to a_p, a_{p+1}, &c., vanish identically, and it is necessary to proceed to a second differentiation. In this case the common roots are given as the roots of the quadratic equation

$$\frac{d^2R}{da_p^{\,2}}x^2 - 2\frac{d^2R}{da_p\,da_{p+1}}x + \frac{d^2R}{da_{p+1}^{\,2}} = 0,$$

as is easily seen by differentiating the value of R above given, when the first member of the equation last written is found to be equal to

$$\left(\frac{d^2\phi}{da_p^{\,2}}x^2 - 2\frac{d^2\phi}{da_p da_{p+1}}x + \frac{d^2\phi}{da_{p+1}^{\,2}}\right)U + \left(\frac{d^2\psi}{da_p^{\,2}}x^2 - 2\frac{d^2\psi}{da_p da_{p+1}}x + \frac{d^2\psi}{da_{p+1}^{\,2}}\right)V,$$

an expression which vanishes when either of the common roots is substituted for x.

A similar process will apply if there are three or more common roots.

The examples which follow are given to illustrate the principles contained in the foregoing chapter.

Examples.

1. Eliminate x from the equations

$$ax^2 + bx + c = 0,$$
$$x^3 = 1.$$

Multiplying the first equation by x, we have, since $x^3 = 1$,

$$bx^2 + cx + a = 0;$$

and multiplying again by x, we have

$$cx^2 + ax + b = 0.$$

Eliminating x^2 and x linearly from these three equations, the result is expressed as a determinant

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0.$$

If the method of symmetric functions (Art. 151) be employed, and the roots of the second equation substituted in the first, the resultant is obtained in the form

$$(a + b + c)(a\omega^2 + b\omega + c)(a\omega + b\omega^2 + c).$$

2. Eliminate similarly x from the equations

$$ax^4 + bx^3 + cx^2 + dx + e = 0, \quad x^5 = 1.$$

The result is a circulant of the fifth order, obtained by a process similar to that of the last example. By aid of the method of symmetric functions the five factors can be written down. An analogous process may be applied in general to any two equations of this kind.

3. Apply the method of Art. 153 to find the conditions that the two cubics

$$\phi(x) = ax^3 + bx^2 + cx + d = 0,$$
$$\psi(x) = a'x^3 + b'x^2 + c'x + d' = 0$$

should have two common roots.

When this is the case, identical results must be obtained by multiplying $\phi(x)$ by the third factor of $\psi(x)$, and $\psi(x)$ by the third factor of $\phi(x)$. We have, therefore,

$$(\lambda' x + \mu')\phi(x) \equiv (\lambda x + \mu)\psi(x),$$

where λ, μ, λ', μ' are indeterminate quantities. This identity leads to the equations

$$\begin{aligned} \lambda' a \qquad - \lambda a' \qquad &= 0, \\ \lambda' b + \mu' a - \lambda b' - \mu a' &= 0, \\ \lambda' c + \mu' b - \lambda c' - \mu b' &= 0, \\ \lambda' d + \mu' c - \lambda d' - \mu c' &= 0, \\ \mu' d \qquad - \mu d' &= 0. \end{aligned}$$

Eliminating λ', μ', λ, μ from every four of these, we obtain five determinants, whose vanishing expresses the required conditions. There is a convenient notation in use to express the result of eliminating from a number of equations of this kind. In the present instance the vanishing of the five determinants is expressed as follows:—

$$\left\| \begin{matrix} a & b & c & d & 0 \\ 0 & a & b & c & d \\ a' & b' & c' & d' & 0 \\ 0 & a' & b' & c' & d' \end{matrix} \right\| = 0,$$

the determinants being formed by omitting each column in turn. It should be observed that the conditions here obtained are equivalent to two independent conditions only, and it can be shown that, when any two of the determinants vanish, the remaining three must vanish also.

4. Prove the identity

$$\begin{vmatrix} \alpha^2 & 2\alpha\beta & \beta^2 \\ \alpha\alpha' & \alpha\beta' + \alpha'\beta & \beta\beta' \\ \alpha'^2 & 2\alpha'\beta' & \beta'^2 \end{vmatrix} \equiv (\alpha\beta' - \alpha'\beta)^3.$$

This appears by eliminating x and y from the equations

$$\alpha x + \beta y = 0, \qquad \alpha' x + \beta' y = 0;$$

for from these equations we derive

$$(\alpha x + \beta y)^2 = 0, \quad (\alpha x + \beta y)(\alpha' x + \beta' y) = 0, \quad (\alpha' x + \beta' y)^2 = 0.$$

The determinant above written is the result of eliminating x^2, xy, and y^2 from the latter equations; and this result must be a power of the determinant derived by eliminating x, y from the linear equations.

5. Prove similarly

$$\begin{vmatrix} \alpha^3 & 3\alpha^2\beta & 3\alpha\beta^2 & \beta^3 \\ \alpha^2\alpha' & \alpha^2\beta' + 2\alpha\alpha'\beta & 2\alpha\beta\beta' + \alpha'\beta^2 & \beta^2\beta' \\ \alpha\alpha'^2 & \alpha'^2\beta + 2\alpha\alpha'\beta' & 2\alpha'\beta\beta' + \alpha\beta'^2 & \beta\beta'^2 \\ \alpha'^3 & 3\alpha'^2\beta' & 3\alpha'\beta'^2 & \beta'^3 \end{vmatrix} \equiv (\alpha\beta' - \alpha'\beta)^6.$$

6. Prove the result of Ex. 13, p. 55, by eliminating λ, μ, λ', μ', from four equations

$$\alpha' = \frac{\lambda\alpha + \mu}{\lambda'\alpha + \mu'}, \quad \beta' = \frac{\lambda\beta + \mu}{\lambda'\beta + \mu'}, \text{ \&c.}$$

connecting the variables in homographic transformation.

7. Given

$$U \equiv Au^2 + 2Buv + Cv^2,$$
$$V \equiv A'u^2 + 2B'uv + C'v^2,$$
$$u \equiv ax^2 + 2bxy + cy^2,$$
$$v \equiv a'x^2 + 2b'xy + c'y^2,$$

determine the resultant of U and V considered as functions of x, y.

Since

$$U = A(u - \alpha v)(u - \beta v),$$
$$V = A'(u - \alpha' v)(u - \beta' v),$$

if U and V vanish for common values of x, y, some pair of factors, as $u - \alpha v$ and $u - \alpha' v$, must vanish; whence forming the resultant of $u - \alpha v$ and $u - \alpha' v$ and representing the resultant of u and v by $R(u, v)$, we have

$$R(u - \alpha v, u - \alpha' v) = (\alpha - \alpha')^2 R(u, v);$$

and multiplying all these resultants together, we find

$$R(U_x, V_x) = A^4 A'^4 (\alpha - \alpha')^2 (\beta - \beta')^2 (\alpha - \beta')^2 (\beta - \alpha')^2 \{R(u, v)\}^4,$$

or

$$R(U_x, V_x) = \{R(U, V)\}^2 \{R(u, v)\}^4.$$

8. Prove that the equation whose roots are the differences of the roots of a given equation $f(x) = 0$ may be obtained by eliminating x from the equations

$$f(x) = 0, \quad f'(x) + f''(x)\frac{y}{1 . 2} + f'''(x)\frac{y^2}{1 . 2 . 3} + \&c. = 0;$$

and determine the degree of the equation in y (cf. Art. 44).

9. Eliminate x, y, z from the equations

$$x + y + z = 0,$$

$$ayz + bzx + cxy = 0,$$

$$ay^3z^3 + bz^3x^3 + cx^3y^3 = 0.$$

Taking the first two equations along with an assumed linear equation with arbitrary coefficients, viz.,

$$\lambda x + \mu y + \nu z = 0,$$

and eliminating x, y, z, we easily obtain

$$a\lambda^2 + b\mu^2 + c\nu^2 + (a - b - c)\mu\nu + (b - c - a)\nu\lambda + (c - a - b)\lambda\mu = 0, \quad (1)$$

which must be equivalent to the equation

$$(\lambda x_1 + \mu y_1 + \nu z_1)(\lambda x_2 + \mu y_2 + \nu z_2) = 0, \quad (2)$$

where $x_1, y_1, z_1, x_2, y_2, z_2$ are the two systems of values of x, y, z common to the first two of the given equations. Substituting these values in the third of the given equations, we have

$$R = (ay_1^3z_1^3 + bz_1^3x_1^3 + cx_1^3y_1^3)(ay_2^3z_2^3 + bz_2^3x_2^3 + cx_2^3y_2^3);$$

and reducing this value of R by means of the symmetric functions determined by the comparison of the equations (1) and (2), we find

$$4R = 4p^2q + q^2 + 27pr,$$

where

$$p = a^2 + b^2 + c^2 - 2bc - 2ca - 2ab,$$

$$q = abc(a + b + c),$$

$$r = a^2b^2c^2.$$

10. If U, V, W are three given functions of x of the degrees $m, n, m + n - 1$, respectively, prove that an identical relation exists of the form

$$RW = U\phi(x) + V\psi(x),$$

where $\phi(x)$ and $\psi(x)$ are functions to be determined, of the degrees $n - 1$ and $m - 1$, respectively, and R is the resultant of U and V.

11. Verify the results of Art. 158 by differentiating the value of R given in Art. 151.

CHAPTER XV.

CALCULATION OF SYMMETRIC FUNCTIONS. SEMINVARIANTS AND SEMICOVARIANTS.

159. Waring's General Expressions for s_m and p_m.—The most fundamental properties of symmetric functions of the roots of equations have been already discussed (Arts. 27, 28, and Chap. VIII., Vol. I.). In the present chapter we add some miscellaneous propositions which may often be used with advantage in the calculation of symmetric functions. The general expressions, due to Waring, referred to in Art. 80, will first be given:—

(1) *General expression for s_m in terms of the coefficients $p_1, p_2, \ldots p_n$ of an equation of the n^{th} degree.*

We have

$$-\log_e(1 + p_1y + \ldots + p_ny^n) = \sum_{r=1}^{r=\infty} \frac{(-1)^r}{r}(p_1y + p_2y^2 + \ldots + p_ny^n)^r$$

$$= s_1y + \frac{1}{2}s_2y^2 + \frac{1}{3}s_3y^3 + \ldots + \frac{1}{m}s_my^m + \ldots \text{ (Art. 79).}$$

Now, making use of the known form of the coefficient of y^m in the expansion of $(p_1y + p_2y^2 + \ldots + p_ny^n)^r$ by the multinomial theorem, and comparing coefficients of y^m in the above equation, we find

$$s_m = \sum \frac{(-1)^r m\Gamma(r_1 + r_2 + \ldots + r_n)}{\Gamma(r_1+1)\,\Gamma(r_2+1)\ldots\Gamma(r_n+1)} p_1^{r_1} p_2^{r_2} \ldots p_n^{r_n},$$

in which $$r_1 + r_2 + r_3 + \ldots + r_n = r,$$

$$r_1 + 2r_2 + 3r_3 + \ldots + nr_n = m;$$

and $r_1, r_2, r_3, \ldots r_n$ are to be given all positive integer values.

zero included, which satisfy the last of these two equations. Also, representing by r_i any of these integers,

$$\Gamma(r_i + 1) = 1 \,.\, 2 \,.\, 3 \ldots r_i,$$

with the assumption that $\Gamma(1) = 1$ when $r_i = 0$.

(2) *General expression for any coefficient* p_m *in terms of the sums of the powers of the roots* $s_1, s_2, \ldots s_m$.

We have

$$1 + p_1 y + \ldots + p_m y^m + \ldots + p_n y^n = e^{-ys_1} \,.\, e^{-\frac{1}{2}y^2 s_2} \,.\, e^{-\frac{1}{3}y^3 s_3} \ldots$$

(Art. 80).

When the factors on the right-hand side of this equation are developed, and the coefficients of y^m on both sides compared, we find, employing the notation of the last example,

$$p_m = \sum \frac{(-1)^{r_1+r_2+\ldots+r_m} s_1^{r_1} s_2^{r_2} \ldots s_m^{r_m}}{\Gamma(r_1+1)\,\Gamma(r_2+1)\ldots\Gamma(r_m+1)\,2^{r_2}3^{r_3}\ldots m^{r_m}},$$

in which $r_1, r_2, \ldots r_m$ are to be given all positive values, zero included, which satisfy the equation

$$r_1 + 2r_2 + 3r_3 + \ldots + mr_m = m.$$

160. **Symmetric Functions of the Roots of two Equations.**—If it be required to calculate a symmetric function involving the roots $a_1, a_2, a_3 \ldots a_m$ of the equation

$$\phi(x) \equiv a_0 x^m + a_1 x^{m-1} + a_2 x^{m-2} + \ldots + a_m = 0, \qquad (1)$$

along with the roots $\beta_1, \beta_2, \beta_3, \ldots \beta_n$, of the equation

$$\psi(y) \equiv b_0 y^n + b_1 y^{n-1} + b_2 y^{n-2} + \ldots + b_n = 0, \qquad (2)$$

we proceed as follows :—

Assume a new variable t connected with x and y by the equation

$$t = \lambda x + \mu y;$$

and let y be eliminated by means of this equation from (2). The result is an equation of the n^{th} degree in x whose coefficients involve λ, μ, and t in the n^{th} power. Now let x be eliminated by any of the preceding methods from this equation and (1). We obtain an equation of the mn^{th} degree in t, whose roots are the mn values of the expression $\lambda\alpha + \mu\beta$.

If, now, it be required to calculate in terms of the coefficients of $\phi(x)$ and $\psi(y)$ any symmetric function such as $\Sigma\alpha^p\beta^q$, we form the sum of the $(p+q)^{th}$ powers of the roots of the equation in t. We thus find the value of $\Sigma(\lambda\alpha + \mu\beta)^{p+q}$ expressed in terms of the original coefficients and the several powers of λ and μ. The coefficient of $\lambda^p\mu^q$ in this expression will furnish the required value of $\Sigma\alpha^p\beta^q$ in terms of the coefficients of $\phi(x)$ and $\psi(y)$.

If it were required to calculate symmetric functions of the roots of three equations, we should assume

$$t = \lambda x + \mu y + \nu z,$$

eliminate x, y, z, and proceed as before. This method therefore applies whatever the number of equations; and by making the coefficients $a_r = b_r = c_r$, &c., we fall back on the symmetric functions of the roots of a single equation already calculated.

161. **Calculation by Sums of Powers of Roots.**—By aid of the following differential equation, connecting a function of the coefficients and its value in terms of the sums of the powers, symmetric functions can often be calculated with great facility:—

$$\frac{d}{ds_r} F(p_1, p_2, \ldots p_n) = -\frac{1}{r}\left(\frac{dF}{dp_r} + p_1\frac{dF}{dp_{r+1}} + \ldots + p_{n-r}\frac{dF}{dp_n}\right).$$

To prove this equation, we take the equation (1) of Art. 80, and differentiate it with regard to s_r. Comparing coefficients of the different powers of y, we have

$$\frac{dp_q}{ds_r} = 0, \text{ when } q < r; \quad \frac{dp_r}{ds_r} = -\frac{1}{r}; \quad \frac{dp_{r+k}}{ds_r} = -\frac{1}{r}p_k;$$

and substituting these values in

$$\frac{d}{ds_r} F(p_1, p_2, \ldots p_n) = \frac{dF}{dp_1}\frac{dp_1}{ds_r} + \frac{dF}{dp_2}\frac{dp_2}{ds_r} + \ldots + \frac{dF}{dp_n}\frac{dp_n}{ds_r},$$

we have at once the equation above written.

Examples.

1. Calculate the value of the symmetric function $\Sigma\alpha_1^2\alpha_2^2\alpha_3^2\alpha_4^2$ of the roots of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n = 0.$$

Knowing the order and weight of any symmetric function, we can write down the literal part of its value in terms of the coefficients. Here Σ is of the second order and its weight is eight; hence

$$\Sigma = t_0p_8 + t_1p_7p_1 + t_2p_6p_2 + t_3p_5p_3 + t_4p_4^2,$$

where t_0, t_1, t_2, &c., are numerical coefficients to be determined.

Terms such as $p_6p_1^2$, $p_5p_1p_2$, $p_5p_1^3$, &c., although of the right weight, are of too high an order, and therefore cannot enter into the expression for Σ. Again, Σ expressed in terms of the sums of the powers of the roots is of the form $F(s_2, s_4, s_6, s_8)$; for, in general, $\Sigma\alpha_1^p\alpha_2^q\alpha_3^r \ldots$, when so expressed, is made up of terms such as s_p, s_{p+q}, s_{p+q+r}, $\ldots s_{kp}$, $\ldots$ all of which are sums of even powers when p, q, r, $\ldots$ are even; therefore in this case none but even sums of powers enter into the expression for Σ.

Also, since $\dfrac{\partial\Sigma}{\partial s_3} = 0$, and $\dfrac{\partial\Sigma}{\partial s_7} = 0$, we have, using the formula above given for $\dfrac{\partial F}{\partial s_r}$,

$$t_0p_5 + t_1p_1p_4 + t_2p_3p_2 + t_3(p_2p_3 + p_5) + 2t_4p_1p_4 = 0,$$

$$t_0p_1 + t_1p_1 = 0.$$

From these equations we infer

$$t_0 + t_1 = 0, \quad t_2 + t_3 = 0, \quad t_3 + t_0 = 0, \quad t_1 + 2t_4 = 0;$$

but $t_4 = 1$, since for a quartic $\Sigma = p_4^2$; therefore

$$t_1 = -2, \quad t_0 = 2, \quad t_3 = -2, \quad t_2 = 2;$$

and, substituting these values of t_0, t_1, t_2, t_3, t_4,

$$\Sigma\alpha_1^2\alpha_2^2\alpha_3^2\alpha_4^2 = 2p_8 - 2p_7p_1 + 2p_6p_2 - 2p_5p_3 + p_4^2.$$

2. Calculate $\Sigma\alpha_1^2\alpha_2^2\alpha_3^2$ for the same equation.

Ans. $-2p_6 + 2p_1p_5 - 2p_2p_4 + p_3^2$. (Cf. Ex. 6, Art. 82.)

3. Calculate for the same equation the symmetric function $\Sigma\alpha_1^3\alpha_2^2\alpha_3$.

Here the weight is six, and the order three; hence

$$\Sigma\alpha_1^3\alpha_2^2\alpha_3 = t_0p_6 + t_1p_5p_1 + t_2p_4p_2 + t_3p_4p_1^2 + t_4p_3^2 + t_5p_1p_2p_3 + t_6p_2^3.$$

Also Σ, expressed in terms of s_1, s_2, s_3, &c., is (Art. 78)

$$s_1s_2s_3 - s_1s_5 - s_3^2 - s_2s_4 + 2s_6.$$

Now, differentiating these two values of Σ with regard to s_6, and comparing differential coefficients, we have

$$t_0\frac{\partial p_6}{\partial s_6} = -\frac{t_0}{6} = 2, \quad \text{or} \quad t_0 = -12.$$

Differentiating with regard to s_5, we have

$$t_0 p_1 + t_1 p_1 = 5s_1 = -5p_1; \quad \therefore\ t_1 = 7.$$

Differentiating with regard to s_4,

$$t_0 p_2 + t_1 p_1^2 + t_2 p_2 + t_3 p_1^2 = 4s_2 = 4(p_1^2 - 2p_2);$$

whence

$$t_0 + t_2 = -8, \quad t_1 + t_3 = 4;$$

and

$$t_3 = -3, \quad t_2 = 4.$$

Again, $t_6 = 0$; for Σ vanishes if $n - 2$ roots vanish. And we find t_4 and t_5 by taking the particular case when $n - 3$ roots vanish; for in this case

$$\Sigma\alpha_1^3\alpha_2^2\alpha_3 = \alpha_1\alpha_2\alpha_3\Sigma\alpha_1^2\alpha_2 = -p_3(-p_1p_2 + 3p_3) = p_1p_2p_3 - 3p_3^2,$$

and therefore $t_4 = -3$, $t_5 = 1$; whence, finally,

$$\Sigma\alpha_1^3\alpha_2^2\alpha_3 = -12p_6 + 7p_1p_5 + 4p_4p_2 - 3p_4p_1^2 - 3p_3^2 + p_1p_2p_3.$$

162. **Functions of Differences of a Cubic.**—The propositions contained in this and the next following Articles are most useful in the calculation of certain classes of symmetric functions of the roots of cubic and biquadratic equations; they are also of great importance, as will appear in the sequel, in reference to the determination of the number of independent invariants and covariants of these forms.

Prop. I.—*Every rational and integral symmetric function* $\phi(\alpha, \beta, \gamma)$ *of the roots of the equation*

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$$

which involves the differences only of these roots is, when multiplied by a_0^{ϖ}, *expressible in the form* $F(a_0, H, \Delta)$, *or* $GF(a_0, H, \Delta)$, *according as* ϕ *is an even or odd function of the roots,* F *being a rational and integral function of* a_0, H, Δ, *and* ϖ *being the order of* ϕ.

It is first necessary to prove the following Lemma:—*There exists no function of* H *and* Δ *which is divisible by* a_0. For if there were any such function $F_p(H, \Delta)$, then, making a_0 vanish, we should have

$$F_p(H', \Delta') = 0, \text{ where } H' = -a_1^2, \ \Delta' = 4a_1^3a_3 - 3a_1^2a_2^2,$$

the values of H and Δ when a_0 vanishes (Art. 42). This equation is clearly impossible; for if we eliminate a_1 by means of the equation $H' = -a_1^2$, the resulting equation will contain a_2 and a_3 as well as H' and Δ'.

To proceed with the proof of the Proposition:—Since ϕ is a function of the differences, we can suppose it to be calculated from the cubic deprived of its second term (Art. 36). We have therefore

$$a_0^r\phi(\alpha, \beta, \gamma) = F(a_0, H, G),$$

in which F is a rational integral function, and r, which cannot be less than ϖ (Art. 81), remains to be determined. Arranging the right-hand side according to powers of G, we may write

$$a_0^r\phi(\alpha, \beta, \gamma) = F_0(a_0, H) + GF_1(a_0, H) + G^2F_2(a_0, H) + \ldots$$

Since the weight of H is even, it follows that when ϕ is an even function of the roots (*i.e.* its weight even), all terms involving odd powers of G must disappear, and when ϕ is an odd function, F_0 and all terms involving even powers of G must disappear. Taking out G as a factor in the latter case, and eliminating even powers of G by means of the relation

$$G^2 + 4H^3 = a_0^2\Delta, \qquad \text{(Art. 42)}$$

we have proved that $a_0^r\phi$ is expressible in the form $F(a_0, H, \Delta)$, or $GF(a_0, H, \Delta)$, according as ϕ is even or odd.

It appears therefore that every odd function of the roots of the kind here considered must have as a factor

$$(2\alpha - \beta - \gamma)(2\beta - \gamma - \alpha)(2\gamma - \alpha - \beta). \quad \text{(Ex. 15, Art. 27.)}$$

We can suppose this factor removed from ϕ, with the corresponding value in terms of the coefficients from the second side of the equation; and it only remains to determine the value of r in the case of an even function of the roots. Writing the relation in the form

$$a_0^r\phi(\alpha, \beta, \gamma) = F(a_0, H, \Delta),$$

arranging the right-hand side according to powers of a_0, and dividing by $a_0^{r-\varpi}$, we have

$$a_0^{\varpi}\phi(\alpha, \beta, \gamma) = F_0(a_0, H, \Delta) + \Sigma\frac{F_p(H, \Delta)}{a_0^p},$$

where F_0 is an integral function of a_0, H, Δ, and Σ contains all the fractional terms. Now, ϕ being a symmetric function whose order is ϖ, $a_0^{\varpi}\phi$ is expressible as an integral function of the coefficients (Art. 81); and since, by the lemma above established, none of the terms included in Σ can become integral, the fractional part must disappear, and the equation assumes the form

$$a_0^{\varpi}\phi(\alpha, \beta, \gamma) = F_0(a_0, H, \Delta).$$

The proposition is therefore proved.

163. **Functions of Differences of a Biquadratic.**—The corresponding proposition for a biquadratic is as follows:—

PROP. II.—*Every rational and integral symmetric function* $\phi(\alpha, \beta, \gamma, \delta)$ *of the roots of the equation*

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0,$$

which involves the differences only of the roots, is, when multiplied by a_0^{ϖ}, *expressible in the form* $F(a_0, H, I, J)$ or $GF(a_0, H, I, J)$ *according as* ϕ *is an even or odd function of the roots,* F *being a rational and integral function of* a_0, H, I, J, *and* ϖ *the order of* ϕ.

The following lemma must first be proved:—*There exists no function of* H, I, J *which is divisible by* a_0. For, suppose if possible $F_p(H, I, J)$ to be such a function. Making a_0 vanish, we have $F_p(H', I', J') = 0$, where $H' = -a_1^2$, $I' = -4a_1a_3 + 3a_2^2$, $J' = 2a_1a_2a_3 - a_4a_1^2 - a_2^3$ (the values of H, I, J, when $a_0 = 0$); but no such relation can exist, since it is impossible to eliminate a_1, a_2, a_3, a_4, so as to obtain a relation between H', I', J' alone.

Now since, as in the preceding Article, ϕ is a function of the differences of the roots, we can suppose it calculated from the equation deprived of its second term (Art. 37). We have therefore

$$a_0^r\phi(\alpha, \beta, \gamma, \delta) = F(a_0, H, I, G),$$

in which F is a rational and integral function, and r remains to be determined. Proceeding as before,

$$a_0^r\phi(\alpha, \beta, \gamma, \delta) = F_0(a_0, H, I) + GF_1(a_0, H, I) + G^2F_2(a_0, H, I) + \ldots$$

Since the weight is even in the case of both the functions H and I, we infer, just as in the preceding Article, that G is a factor in the odd functions; and, eliminating even functions of G by the relation

$$G^2 = a_0^2(HI - a_0J) - 4H^3, \qquad \text{(Art. 37)}$$

we prove that $a_0^r\phi$ is expressible in the form $F(a_0, H, I, J)$ or $GF(a_0, H, I, J)$ according as ϕ is even or odd. It appears, therefore, that every odd function of the roots of the kind here considered contains the factor

$$(\beta + \gamma - \alpha - \delta)(\gamma + \alpha - \beta - \delta)(\alpha + \beta - \gamma - \delta).$$

(Ex. 20, Art. 27.)

Removing this factor, we proceed to determine r in the case of an even function. Writing the relation in the form

$$a_0^r\phi(\alpha, \beta, \gamma, \delta) = F(a_0, H, I, J),$$

and dividing by $a_0^{r-\varpi}$, we have, as in the preceding Article,

$$a_0^\varpi\phi(\alpha, \beta, \gamma, \delta) = F_0(a_0, H, I, J) + \Sigma\frac{F_p(H, I, J)}{a_0^p}.$$

Now since the right-hand side must be an integral function of the coefficients (Art. 81), and since, by the lemma above established, none of the terms included in Σ can become integral, we have

$$a_0^\varpi\phi(\alpha, \beta, \gamma, \delta) = F_0(a_0, H, I, J),$$

which proves the proposition.

Instances of the use of this proposition in the calculation of symmetric functions of the roots of a biquadratic will be found among the examples at the end of the chapter.

164. **Seminvariants and Semicovariants.** — Let $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$ be the roots of

$$a_0x^n + na_1x^{n-1} + \frac{n(n-1)}{1 \cdot 2}a_2x^{n-2} + \ldots + a_n = 0,$$

the general equation written with binomial coefficients. We proceed to the consideration of an important class of functions of x which may be derived from a given symmetric function of the roots.

In the two preceding Articles we have been occupied with certain special kinds of homogeneous symmetric functions of the roots which contain the *differences only* of these quantities (cf. Art. 36). Such functions may be called (for a reason which will appear in a subsequent chapter) *semi-invariants*, or, as it is usually written, *seminvariants*. Being symmetric functions of the roots, they are expressible (when multiplied by a power of a_0) in a rational and integral form in terms of the coefficients.

We may use in like manner the term *semicovariants* to denote similar functions of the differences of the quantities $x, \alpha_1, \alpha_2, \ldots \alpha_n$, such that, when they are arranged in powers of x, the successive coefficients of x are expressible in a similar manner in terms of the coefficients.

We proceed now to show how semicovariants may be generated, and then expanded in powers of x, when expressed either in terms of the roots or in terms of the coefficients.

From any relation such as

$$a_0^{\varpi}\phi(\alpha_1, \alpha_2, \ldots \alpha_n) = F(a_0, a_1, a_2, \ldots a_n),$$

where ϕ is an integral function of the order ϖ, and F the corresponding expression in terms of the coefficients, we may, by diminishing each of the roots by x, and consequently changing any coefficient a_r into U_r (see Art. 35), derive the following equation:—

$$a_0^{\varpi}\phi(\alpha_1 - x,\ \alpha_2 - x, \ldots \alpha_n - x) = F(U_0, U_1, U_2, \ldots U_n), \qquad (1)$$

thus obtaining two forms for a semicovariant, one expressed in terms of the roots, and the other in terms of the coefficients.

To expand these forms in powers of x, we have, for the first member of the equation, by Taylor's theorem,

$$\phi(\alpha_1 - x,\ \alpha_2 - x, \ldots \alpha_n - x) = \phi_0 + x\delta\phi_0 + \frac{x^2}{1.2}\delta^2\phi_0 + \ldots. \qquad (2)$$

where
$$\phi_0 = \phi(a_1, a_2, \ldots a_n),$$

and
$$-\delta \equiv \frac{\partial}{\partial\alpha_1} + \frac{\partial}{\partial\alpha_2} + \ldots + \frac{\partial}{\partial\alpha_n}.$$

Again, omitting all powers of x higher than the first, the second member of the equation becomes

$$F(a_0, a_1 + a_0x, a_2 + 2a_1x, \ldots a_n + na_{n-1}x),$$

or, when expanded,
$$F_0 + xDF_0 + \&c.,$$
where
$$F_0 = F(a_0, a_1, a_2, \ldots a_n),$$

and
$$D \equiv a_0\frac{\partial}{\partial a_1} + 2a_1\frac{\partial}{\partial a_2} + 3a_2\frac{\partial}{\partial a_3} + \ldots + na_{n-1}\frac{\partial}{\partial a_n}.$$

Comparing the two expanded forms, we have

$$a_0^{\pi}\delta\phi(a_1, a_2, \ldots a_n) = DF(a_0, a_1, \ldots a_n),$$

and consequently, by successive applications of the operators δ and D,
$$a_0^{\pi}\delta^r\phi(a_1, a_2, \ldots a_n) = D^rF(a_0, a_1, \ldots a_n);$$

whence we infer from the expansion (2)

$$F(U_0, U_1, \ldots U_n) = F_0 + xDF_0 + \frac{x^2}{1.2}D^2F_0 + \&c. \ldots .$$

By the aid, therefore, of the two operators—δ in terms of the roots, and D in terms of the coefficients—we can expand at pleasure either side of the equation (1) in powers of x. By means of the successive operations of δ we obtain a series of functions of the roots; and, by means of D, their equivalent values in terms of the coefficients.

The results now arrived at are equally true if the function ϕ involves the roots of two or more equations, F being the corresponding value in terms of the coefficients of these equations, and D and δ being replaced by the sums of the similar operators relative to each equation.

It is important to observe that when $\delta\phi_0$ vanishes identically, so also

$$\delta(\delta\phi_0) \text{ or } \delta^2\phi_0 = 0, \quad \delta^3\phi_0 = 0, \text{ \&c.},$$

and therefore x disappears in the expansion of the first member of equation (1). Now this can happen only when ϕ is a function of the differences of $a_1, a_2, \ldots a_n$; whence we conclude that if $F(a_0, a_1, \ldots a_n)$ is a seminvariant

$$DF(a_0, a_1, a_2, \ldots a_n) = 0.$$

This identical relation is often sufficient to determine the numerical coefficients in a seminvariant when the order and weight are known. If there should be two or more seminvariants of the same order and weight, the operation of D will not supply equations enough to determine all the assumed coefficients, as will appear from the discussion in the next Article. If no seminvariant exists of the required order and weight, the coefficients will all vanish.

165. **Determination of Seminvariants.**—The problem of finding the seminvariants of a given order ϖ and weight κ of a quantic is the same as that of determining all such solutions of the differential equation

$$D\Phi \equiv a_0 \frac{\partial\Phi}{\partial a_1} + 2a_1 \frac{\partial\Phi}{\partial a_2} + \ldots + na_{n-1} \frac{\partial\Phi}{\partial a_n} = 0. \qquad (1)$$

To solve this equation when possible, assume

$$\Phi = \lambda_1\phi_1 + \lambda_2\phi_2 + \ldots + \lambda_r\phi_r, \qquad (2)$$

where $\phi_1, \phi_2, \ldots \phi_r$ are all the possible combinations of $a_0, a_1, a_2, \ldots a_n$ of the order ϖ and weight κ, and $\lambda_1, \lambda_2, \ldots \lambda_r$ arbitrary multipliers.

Now, substituting this value of Φ in the equation $D\Phi = 0$, we have as the result

$$L_1\psi_1 + L_2\psi_2 + \ldots + L_p\psi_p = 0,$$

where $\psi_1, \psi_2, \psi_3, \ldots \psi_p$ are all the distinct terms of the order ϖ and weight $\kappa - 1$, and $L_1, L_2, \ldots L_p$ are linear functions of $\lambda_1, \lambda_2, \ldots \lambda_p$, which must all vanish when Φ is a seminvariant.

To determine $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_r,$ we have

$$\left.\begin{array}{l} L_1 = l_{11}\lambda_1 + l_{12}\lambda_2 + \ldots + l_{1r}\lambda_r = 0, \\ L_2 = l_{21}\lambda_1 + l_{22}\lambda_2 + \ldots + l_{2r}\lambda_r = 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ L_p = l_{p1}\lambda_1 + l_{p2}\lambda_2 + \ldots + l_{pr}\lambda_r = 0 \end{array}\right\}. \qquad (3)$$

There are three distinct cases to be now considered:—

(1). When r is greater than p, there are not sufficient equations to determine all the quantities $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_r$; but any p of them can be determined as linear functions of the rest. For this purpose we can proceed as follows:—Introduce $r-p=j$ arbitrary multipliers defined by the equations

$$\left.\begin{array}{l} m_{11}\lambda_1 + m_{12}\lambda_2 + \ldots + m_{1r}\lambda_r = \Lambda_1, \\ m_{21}\lambda_1 + m_{22}\lambda_2 + \ldots + m_{2r}\lambda_r = \Lambda_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ m_{j1}\lambda_1 + m_{j2}\lambda_2 + \ldots + m_{jr}\lambda_r = \Lambda_j \end{array}\right\}. \qquad (4)$$

Solving the equations (3) and (4) for $\lambda_1, \lambda_2, \ldots \lambda_r$, and substituting in equation (2), we have the following value for Φ:—

$$\Delta\Phi = \Lambda_1\Sigma_1 + \Lambda_2\Sigma_2 + \Lambda_3\Sigma_3 + \ldots + \Lambda_j\Sigma_j,$$

and therefore

$$\Delta D\Phi = \Lambda_1 D\Sigma_1 + \Lambda_2 D\Sigma_2 + \ldots + \Lambda_j D\Sigma_j = 0,$$

whence $\qquad D\Sigma_1 = 0, \quad D\Sigma_2 = 0, \ldots D\Sigma_j = 0,$

since $\Lambda_1, \Lambda_2, \ldots \Lambda_j$ may have any values whatever.

We conclude, therefore, that in this case there are $r-p=j$ linearly independent seminvariants.

(2). When p is equal to r or greater than r, the equations

$$L_1 = 0, \quad L_2 = 0, \ldots L_p = 0$$

cannot, in general, be satisfied, and there are no seminvariants of the quantic of the order ϖ and weight κ.

(3). When $p = r - 1$ there are just sufficient equations to determine the ratios of $\lambda_1, \lambda_2, \ldots \lambda_r$, and consequently only one seminvariant exists.

Examples.

1. Determine for a cubic a seminvariant whose order and weight are both three.

Assume
$$\phi = Aa_0^2a_3 + Ba_0a_1a_2 + Ca_1^3,$$
these being the only three terms which satisfy the required conditions. It is evident from the form of D that the operation is performed by applying to the suffix of any coefficient a_r the same process as in ordinary differentiation is applied to the index. Thus $Da_r = ra_{r-1}$, and therefore
$$D\phi = (3A + B)\, a_0^2a_2 + (2B + 3C)\, a_1^2a_0 = 0.$$
Hence
$$3A + B = 0, \quad \text{and} \quad 2B + 3C = 0;$$
and putting $A = 1$, we have $B = -3$, and $C = 2$: whence, finally,
$$\phi = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3 = G. \qquad \text{(See Art. 36.)}$$

For a quadratic no such seminvariant can be formed.

2. Investigate seminvariants of a quartic whose order and weight are both four.

Assuming
$$\phi = Aa_0^3a_4 + Ba_0^2a_1a_3 + Ca_0^2a_2^2 + Da_0a_1^2a_2 + Ea_1^4,$$
we readily find
$$D\phi = (4A + B)\, a_0^3a_3 + (3B + 4C + 2D)\, a_0^2a_1a_2 + (2D + 4E)\, a_0a_1^3.$$

We have now only three equations among the assumed five coefficients, whose ratios cannot consequently be determined completely. Expressing B, C, and D in terms of A and E, we have easily
$$\phi = Aa_0^2(a_0a_4 - 4a_1a_3 + 3a_2^2) + E(a_0^2a_2^2 - 2a_0a_1^2a_2 + a_1^4),$$
viz.,
$$\phi = Aa_0^2I + EH^2,$$
where A and E may have any values. We may say therefore that there are in this case two independent fundamental seminvariants of the required order and weight, viz., a_0^2I and H^2; and from these may be derived an indefinite number of seminvariants of the same order and weight by assigning to A and E different numerical values.

3. Determine for a cubic a seminvariant whose order is four and weight six.

Assume
$$\phi = Aa_0^2a_3^2 + Ba_0a_2^3 + Ca_3a_1^3 + Da_1^2a_2^2 + Ea_0a_1a_2a_3,$$
whence
$$D\phi = (6A + E)\, a_0^2a_2a_3 + (6B + 3E + 2D)\, a_0a_1a_2^2 + (3C + 4D)\, a_1^3a_2 + (3C + 2E)\, a_0a_1^2a_3 = 0.$$

Now let $A = 1$, whence $E = -6$; also $3C + 2E = 0$, giving $C = 4$; and $3C + 4D = 0$, giving $D = -3$; and from $6B + 3E + 2D = 0$, we have finally $B = 4$.

Hence $$\phi = a_0^2a_3^2 + 4a_0a_2^3 + 4a_3a_1^3 - 3a_1^2a_2^2 - 6a_0a_1a_2a_3.$$

Compare Art. 42, where the value of ϕ is given in terms of the roots.

4. Determine a seminvariant of a quintic whose order is three and weight five.

It is easily seen that the only terms of the required order and weight are $a_0^2a_5$, $a_0a_1a_4$, $a_0a_2a_3$, $a_1^2a_3$, and $a_1a_2^2$. Proceeding as before we find that the ratios of the assumed coefficients are determinate, and the seminvariant is found to be

$$a_0^2a_5 - 5a_0a_1a_4 + 2a_0a_2a_3 - 6a_1a_2^2 + 8a_1^2a_3.$$

5. Determine for a quartic a seminvariant whose order is three and weight six.

Ans. $a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 - a_2^3 \equiv J.$

6. Investigate for the general equation the seminvariants whose order is three and weight six.

It is easily seen that the only terms which can enter into such seminvariants additional to those which occur in the preceding example are $a_0^2a_6$ and $a_0a_1a_5$. Writing down the function ϕ consisting of seven terms with indeterminate coefficients, and applying the operator D, we find that there are only five equations among the assumed coefficients. We obtain therefore, as is easily seen, seminvariants of the form

$$\lambda a_0(a_0a_6 - 6a_1a_5 + 15a_2a_4 - 10a_3^2) + \mu J,$$

in which λ and μ remain undetermined, their multipliers in this expression being two fundamental seminvariants of the required type.

It may be observed that $a_0a_6 - 6a_1a_5 + 15a_2a_4 - 10a_3^2$ is an invariant of a sextic. This function can be readily found directly by investigating seminvariants whose order is two and weight six. Invariants being, as well as seminvariants, symmetric functions of the roots which contain the differences only are obtained by the present method of investigation; and any function of the coefficients so obtained which is an invariant for a quantic of one particular order will be a seminvariant for quantics (written with binomial coefficients) of all higher orders. The function obtained in Ex. 3 is an invariant of a cubic, and J is an invariant of a quartic. It must be carefully noted, however, that most seminvariants, as *e.g.* those obtained in Exs. 1, 4, are not invariants for quantics of any degree, as will be seen from the definition of an invariant and its properties discussed in the next chapter.

7. Investigate for a quartic seminvariants of order four and weight six.

The only terms additional to those of Ex. 3 are $a_0^2a_2a_4$ and $a_0a_4a_1^2$. Adding therefore $\lambda a_0^2a_2a_4 + \mu a_0a_4a_1^2$ to the value of ϕ in Ex. 3, and operating by D, we find, after expressing the remaining coefficients in terms of λ and A, the following value of ϕ,

$$\phi = \lambda(a_0^2a_2a_4 - a_0a_4a_1^2 + 3a_0a_2^3 + 4a_3a_1^3 - 3a_1^2a_2^2 - 4a_0a_1a_2a_3) + A\Delta_3,$$

where Δ_3 is the function obtained in Ex. 3, viz. the discriminant of the cubic.

Observing that the multiplier of λ is the product of the functions H and I, and substituting for Δ_3 its value $HI - a_0J$ (Art. 42), we have

$$\phi = \lambda' HI + \mu' a_0 J.$$

For a quartic, therefore, the functions HI and a_0J are two fundamental seminvariants of the required order and weight.

8. Investigate seminvariants of the same order and weight as in Ex. 7 for quantics of the sixth and higher orders.

It will be found that there are in this case two equations less than would be required to determine the ratios of the assumed coefficients, and there will consequently be three fundamental seminvariants. It may be easily shown that all seminvariants of the required type can be represented in the form

$$\phi = \lambda a_0^2 (a_0a_6 - 6a_1a_5 + 15a_2a_4 - 10a_3^2) + \mu HI + \nu a_0 J.$$

9. Prove that any seminvariant of the equation

$$(a_0, a_1, \ldots a_r)(x, 1)^r = 0$$

is also a seminvariant of the equation

$$(a_0, a_1, \ldots a_r \ldots a_n)(x, 1)^n = 0,$$

n being greater than r.

10. Determine a seminvariant of a sextic whose order is three and weight eight.

Ans. $a_0a_2a_6 - a_0a_3a_5 + 2a_0a_4^2 - a_1^2a_6 + 3a_1a_2a_5 - a_1a_3a_4 - 3a_2^2a_4 + 2a_2a_3^2$.

Prof. Cayley originally enunciated the important theorem which forms the subject of the foregoing Article, viz.—that the number of linearly independent seminvariants of order ϖ and weight κ is $r - p$, where r is the number of terms of this order and weight which can be formed from the coefficients $a_0, a_1, \ldots a_n$, and p the number of terms of the same order and weight $\kappa - 1$, which can be formed from the same coefficients. In the discussion above given, it is assumed that $L_1, L_2, \ldots L_p$ are linearly independent; and it should be observed that if certain linear relations connected them, for each such relation the number p would be reduced by one. Cayley himself gave no proof of the independence in general of these quantities; but proofs have been supplied by Sylvester (*Crelle*, vol. 85, p. 89) and by Prof. Elliott (*Algebra of Quantics*, Art. 128 *See also Note F at the end of this Volume*).

EXAMPLES.

1. Prove directly that

$$\frac{d}{dx} F(U_0, U_1, U_2 \dots U_n) = DF(U_0, U_1, U_2 \dots U_n).$$

This follows readily from the equations

$$DU_r = rU_{r-1} = \frac{dU_r}{dx}; \quad D\{F(U_p, U_q \dots)\} = \frac{d}{dx}\{U_p, U_q \dots\}.$$

2. Expand $F(U_0, U_1, U_2, \dots U_n)$ by Maclaurin's theorem; and hence prove

$$F(U_0, U_1, \dots U_n) = F_0 + xDF_0 + \frac{x^2}{1.2} D^2 F_0 + \&c.,$$

where

$$F_0 = F(a_0, a_1, a_2, \dots a_n).$$

3. Determine $\phi_1, \phi_2, \dots \phi_j, \dots \phi_p$ from the equations

$$\begin{aligned}
\phi_1 + \phi_2 + \dots + \phi_p &= T_0,\\
\phi_1\theta_1 + \phi_2\theta_2 + \dots + \phi_p\theta_p &= T_1,\\
\phi_1\theta_1^2 + \phi_2\theta_2^2 + \dots + \phi_p\theta_p^2 &= T_2,\\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
\phi_1\theta_1^{p-1} + \phi_2\theta_2^{p-1} + \dots + \phi_p\theta_p^{p-1} &= T_{p-1}.
\end{aligned}$$

This is an extension of an example already solved (Ex. 1, p. 38); and it will be readily found by applying the method there employed that ϕ_j is given as a function of the $(p-1)^{th}$ degree in θ_j by the equation

$$\begin{vmatrix}
1 & \theta_j & \theta_j^2 & \cdot & \cdot & \theta_j^{p-1} & \phi_j\\
s_0 & s_1 & s_2 & \cdot & \cdot & s_{p-1} & T_0\\
s_1 & s_2 & s_3 & \cdot & \cdot & s_p & T_1\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
s_{p-1} & s_p & s_{p+1} & \cdot & \cdot & s_{2p-2} & T_{p-1}
\end{vmatrix} = 0,$$

where $s_k = \theta_1^k + \theta_2^k + \theta_3^k + \dots + \theta_p^k$.

4. Prove that

$$\Pi = a_0^6(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2(\alpha-\delta)^2(\beta-\delta)^2(\gamma-\delta)^2 = I^3 + mJ^2,$$

where

$$m = -27.$$

We make use of the proposition of Art. 163, and express the given function of the roots, whose order is 6 and weight 12, in terms of a_0, H, I, J. From the table—

	Order.	Weight.
H	2	2
I	2	4
J	3	6

it is easy to see that H cannot enter, for the terms of the sixth order containing H, viz. H^3, H^2I, HI^2, have not the proper weight. Therefore Π must be of the form $lI^3 + mJ^2$, where l and m are numerical coefficients.

Now put a_3 and a_4 equal to zero, and Π will vanish, since in that case the quartic will have equal roots; hence, employing the reduced values of I and J,

$$0 = l\,(3a_2{}^2)^3 + m\,(-a_2{}^3)^2, \quad \text{and therefore} \quad m = -27l.$$

In applying this method to obtain the values of symmetric functions, the rule to be followed in every case is—Retain those terms of weight κ whose order is not greater than ϖ, and make the whole homogeneous by multiplying terms whose order is less than ϖ by suitable powers of a_0.

5. Calculate the symmetric function of the roots of a biquadratic

$$\Sigma\,(\beta-\gamma)^2\,(\gamma-\alpha)^2\,(\alpha-\beta)^2.$$

Since the order of this symmetric function is four and its weight six, we may assume

$$a_0{}^4\Sigma\,(\beta-\gamma)^2\,(\gamma-\alpha)^2\,(\alpha-\beta)^2 = lHI + ma_0J. \qquad (1)$$

The values of l and m may be found by putting $a_3 = 0$, $a_4 = 0$, as in the preceding example, and calculating the value of the reduced symmetric function (when $\gamma = 0$, $\delta = 0$) in terms of the coefficients of the quadratic equation

$$a_0x^2 + 4a_1x + 6a_2 = 0.$$

Identifying then this value with the reduced value of $lHI + ma_0J$, we obtain two simple equations to determine l and m. Or we may proceed as follows by taking two biquadratics whose roots are known, and calculating in each case the symmetric function by actually substituting the roots, and then comparing both sides of the equation when H, I, J are replaced by their values calculated from the numerical coefficients.

First we take the biquadratic equation $6x^4 - 6x^2 = 0$, whose roots are $0, 0, 1, -1$, whence

$$\Sigma = 8, \quad H = -6, \quad I = 3, \quad J = 1.$$

Substituting in equation (1), we have $1728 = -3l + m$.

Proceeding in the same way with the biquadratic equation

we find $x^4 - 6x^2 + 5 = 0$, whose roots are $\pm\sqrt{5},\ \pm 1$,

whence $\Sigma = 768,\quad H = -1,\quad I = 8,\quad J = -4;$

and $-192 = 2l + m,$

and finally, $l = -2 \times 192,\quad m = 3 \times 192;$

$$a_0^4\Sigma = 192\,(-2HI + 3a_0J).$$

6. If $\alpha, \beta, \gamma, \delta$ be the roots of the equation

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0,$$

calculate in terms of a_0, H, I, J the value of the symmetric function

$$a_0^6\Sigma\,(3\alpha - \beta - \gamma - \delta)^2\,(3\beta - \gamma - \delta - \alpha)^2\,(3\gamma - \delta - \alpha - \beta)^2.$$

This may be solved by the same method as the two preceding examples, or we may proceed as follows:—

$$a_0^6\Sigma = 4^6\Sigma z_1^2z_2^2z_3^2,$$

where z_1, z_2, z_3, z_4 are the roots of the equation

$$z^4 + 6Hz^2 + 4Gz + a_0^2I - 3H^2 = 0. \qquad \text{(Art. 37.)}$$

Hence, by Ex. 2, Art. 161,

 Ans. $4^7\{-7H^3 + a_0^2HI - 4a_0^3J\}$.

7. If $F(a_0, a_1, \ldots a_n)$ is a seminvariant of the equation $(a_0, a_1, \ldots a_n)(x, 1)^n = 0$, prove that the same function of the sums of the powers of the roots, viz. $F(s_0, s_1, s_2, \ldots s_n)$ is also a seminvariant. (Mr. M. Roberts.)

This follows by operating on the first function by D, and on the second by $-\delta$, and observing that $Da_r = ra_{r-1}$ and $-\delta s_r = rs_{r-1}$. We thus obtain results identical in form; and if one vanishes identically, so must the other.

8. Calculate the determinant

$$\Delta = \begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix}$$

in terms of the coefficients of a quartic.

By the preceding example, this determinant is a function of the differences of the roots; we may therefore remove the second term of the quartic before calculating it; and if the equation so transformed be

$$y^4 + P_2y^2 + P_3y + P_4 = 0,$$

$$\Delta = \begin{vmatrix} 4 & 0 & -2P_2 \\ 0 & -2P_2 & -3P_3 \\ -2P_2 & -3P_3 & 2P_2^2 - 4P_4 \end{vmatrix} = 4\{8P_2P_4 - 2P_2^3 - 9P_3^2\};$$

but $$a_0^2P_2 = 6H, \quad a_0^3P_3 = 4G, \quad a_0^4P_4 = a_0^2I - 3H^2.$$

Substituting for P_2, P_3, P_4 these values, we have

$$a_0^4\Delta = 192\,(-2HI + 3a_0J):$$

the same result as in Ex. 5 (cf. Ex. 7, p. 35).

9. If $\alpha, \beta, \gamma, \delta$ be the roots of the equation

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0,$$

express H_s, I_s, J_s, G_s of the equation

$$s_0x^4 + 4s_1x^3 + 6s_2x^2 + 4s_3x + s_4 \equiv \Sigma\,(x+\alpha)^4 = 0$$

in terms of H, I, J, G.

$$Ans.\ \frac{H_s}{s_0^2} = -3\,\frac{H}{a_0^2}, \quad \frac{I_s}{s_0^2} = \frac{48H^2 - a_0^2I}{a_0^4}, \quad \frac{G_s}{s_0^3} = -3\,\frac{G}{a_0^3};$$

and by the aid of the relations

$$G^2 + 4H^3 \equiv a_0^2\,(HI - a_0J), \quad G_s^2 + 4H_s^3 \equiv s_0^2\,(H_sI_s - s_0J_s),$$

$$J_s = \frac{192}{a_0^6}\,(3a_0J - 2HI).$$

10. When p is even, prove that

$$\Sigma\,(\alpha_1 - \alpha_2)^p = s_0s_p - ps_1s_{p-1} + \tfrac{1}{2}p\,(p-1)\,s_2s_{p-2} - \&c.$$

Since

$$\Sigma\,(x-\alpha)^p = nx^p - ps_1x^{p-1} + \frac{p\,.\,p-1}{2}\,s_2x^{p-2} - \&c.\ldots - ps_{p-1}x + s_p,$$

changing x into $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$, in succession, and adding the results on both sides of the equations thus obtained, we find

$$2\Sigma\,(\alpha_1 - \alpha_2)^p = s_0s_p - ps_1s_{p-1} + \frac{p\,.\,p-1}{1\,.\,2}\,s_2s_{p-2} - \ldots - ps_1s_{p-1} + s_0s_p,$$

where all the terms on the right side of this equation are repeated except the middle term. Thus

$$\Sigma\,(\alpha_1 - \alpha_2)^4 = s_0s_4 - 4s_1s_3 + 3s_2^2,$$

$$\Sigma\,(\alpha_1 - \alpha_2)^6 = s_0s_6 - 6s_1s_5 + 15s_2s_4 - 10s_3^2, \ \&c.$$

11. Form the equation whose roots are $\phi'(\alpha), \phi'(\beta), \phi'(\gamma), \phi'(\delta)$, where $\alpha, \beta, \gamma, \delta$ are the roots of the equation

$$a_0\phi\,(x) \equiv a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0.$$

$$Ans.\ \phi'^4 + \frac{32G}{a_0^3}\,\phi'^3 + \frac{96\,(2HI - 3a_0J)}{a_0^4}\,\phi'^2 + \frac{256\,(I^3 - 27J^2)}{a_0^6} = 0.$$

12. If $$\Sigma(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2(x-\delta)^4,$$
when expanded, becomes
$$K_0x^4+4K_1x^3+6K_2x^2+4K_3x+K_4;$$
prove that
$$\frac{K_0\alpha\beta\gamma+K_1(\beta\gamma+\gamma\alpha+\alpha\beta)+K_2(\alpha+\beta+\gamma)+K_3}{(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)}=\frac{\pm 16\sqrt{\Delta}}{a_0^3},$$
where $$\Delta=I^3-27J^2.$$

13. Prove that
$$a_0^4\Sigma(\beta+\gamma-\alpha-\delta)^2(\beta-\gamma)^2(\alpha-\delta)^2=192(3a_0J-2HI).$$

14. Prove that
$$a_0^6\Sigma(\beta+\gamma-\alpha-\delta)^4(\beta-\gamma)^2(\alpha-\delta)^2=512(a_0^2I^2-36a_0HJ+12H^2I).$$

15. The quotient of a simple alternant (one, namely, in which each element is a single power) by the difference-product (see Ex. 31, p. 61) can be expressed as a determinant whose elements are the sums of the homogeneous products of the quantities involved.

We take a determinant of the third order, and propose to prove
$$\begin{vmatrix}\alpha^p & \alpha^q & \alpha^r\\ \beta^p & \beta^q & \beta^r\\ \gamma^p & \gamma^q & \gamma^r\end{vmatrix}=\begin{vmatrix}\Pi_p & \Pi_q & \Pi_r\\ \Pi_{p-1} & \Pi_{q-1} & \Pi_{r-1}\\ \Pi_{p-2} & \Pi_{q-2} & \Pi_{r-2}\end{vmatrix}\begin{vmatrix}1 & \alpha & \alpha^2\\ 1 & \beta & \beta^2\\ 1 & \gamma & \gamma^2\end{vmatrix},$$

where Π_p, Π_q, &c., are the sums of the homogeneous products of α, β, γ, as defined in Art. 83, Vol. I. The method employed is perfectly general. Take the following identity, which is easily proved:—
$$\begin{vmatrix}\frac{x}{x-\alpha} & \frac{y}{y-\alpha} & \frac{z}{z-\alpha}\\ \frac{x}{x-\beta} & \frac{y}{y-\beta} & \frac{z}{z-\beta}\\ \frac{x}{x-\gamma} & \frac{y}{y-\gamma} & \frac{z}{z-\gamma}\end{vmatrix}=\frac{\begin{vmatrix}x^3 & y^3 & z^3\\ x^2 & y^2 & z^2\\ x & y & z\end{vmatrix}\begin{vmatrix}1 & \alpha & \alpha^2\\ 1 & \beta & \beta^2\\ 1 & \gamma & \gamma^2\end{vmatrix}}{(x-\alpha)(x-\beta)(x-\gamma)(y-\alpha)(y-\beta)(y-\gamma)(z-\alpha)(z-\beta)(z-\gamma)};$$
write $(x-\alpha)(x-\beta)(x-\gamma)$ as a divisor under each of the elements of the first column on the right-hand side, $(y-\alpha)(y-\beta)(y-\gamma)$ under those of the second, and $(z-\alpha)(z-\beta)(z-\gamma)$ under those of the third, and substitute from the following and similar equations (Ex. 1, Art. 83):—
$$\frac{x}{x-\alpha}=1+\alpha x'+\alpha^2x'^2+\ldots+\alpha^px'^p+\ldots,$$

$$\frac{x^3}{(x-\alpha)(x-\beta)(x-\gamma)} = 1 + \Pi_1 x' + \Pi_2 x'^2 + \ldots + \Pi_p x'^p + \&c.,$$

where $$x' = \frac{1}{x}, \quad y' = \frac{1}{y}, \quad z' = \frac{1}{z}.$$

The identity written above becomes then

$$\begin{vmatrix} 1 + \alpha x' + \ldots + \alpha^p x'^p + \ldots \\ 1 + \beta x' + \ldots + \beta^p x'^p + \ldots \\ 1 + \gamma x' + \ldots + \gamma^p x'^p + \ldots \end{vmatrix} = \begin{vmatrix} 1 + \Pi_1 x' + \Pi_2 x'^2 + \ldots + \Pi_p x'^p + \ldots \\ x' + \Pi_1 x'^2 + \ldots + \Pi_{p-1} x'^p + \ldots \\ x'^2 + \ldots + \Pi_{p-2} x'^p + \ldots \end{vmatrix} \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix};$$

where the second and third columns of the determinants here written can be supplied by replacing x' by y' and z', respectively. Comparing coefficients of $x'^p y'^q z'^r$ on both sides, we have the required result. It should be noticed that when the difference-product determinant is written in the form used above (viz. with ascending powers in the order of the columns), the sign to be attached to the product is always positive, since the product of the two determinants, containing the term $\Pi_p \Pi_{q-1} \Pi_{r-2} \beta \gamma^2$, must contain the term $\alpha^p \beta^q \gamma^r$. Note also, in applying this calculation to particular examples, that $\Pi_0 = 1$, and $\Pi_j = 0$ when j is negative.

16. Prove, by the preceding example,

$$\begin{vmatrix} 1 & \alpha^2 & \alpha^5 \\ 1 & \beta^2 & \beta^5 \\ 1 & \gamma^2 & \gamma^5 \end{vmatrix} = \begin{vmatrix} \Pi_0 & \Pi_2 & \Pi_5 \\ 0 & \Pi_1 & \Pi_4 \\ 0 & \Pi_0 & \Pi_3 \end{vmatrix} \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix}.$$

The quotient, therefore, of the given determinant by the difference-product is $\Pi_1\Pi_3 - \Pi_4$, which may be shown to be equal to $\Sigma\alpha^3\beta + \Sigma\alpha^2\beta^2 + 2\Sigma\alpha^2\beta\gamma$.

17. Prove, by the method of Ex. 15,

$$\begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 \ldots \alpha_1^{n-2} & \alpha_1^m \\ 1 & \alpha_2 & \alpha_2^2 \ldots \alpha_2^{n-2} & \alpha_2^m \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \alpha_n & \alpha_n^2 \ldots \alpha_n^{n-2} & \alpha_n^m \end{vmatrix} = \Pi_{m-n+1} \begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 \ldots \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 \ldots \alpha_2^{n-1} \\ \cdot & \cdot & \cdot \\ 1 & \alpha_n & \alpha_n^2 \ldots \alpha_n^{n-1} \end{vmatrix},$$

where $m =$ or $> n$.

This result may be derived directly from Ex. 1. Art. 83.

CHAPTER XVI.

COVARIANTS AND INVARIANTS.

166. **Definitions.**—In this and the following chapters the notation

$$(a_0, a_1, a_2, \ldots a_n)(x, y)^n$$

will be employed to represent the quantic

$$a_0x^n + na_1x^{n-1}y + \frac{n(n-1)}{1.2}a_2x^{n-2}y^2 + \ldots + na_{n-1}xy^{n-1} + a_ny^n,$$

a homogeneous function of x and y, written with binomial coefficients. If we put $y = 1$, this quantic becomes U_n of Art. 35; and the same notation may be used to denote the homogeneous quantic written in x and y.

Let ϕ be a seminvariant (as defined in the preceding chapter), of the order ϖ, of the roots $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ of the equation $U_n \equiv (a_0, a_1, a_2, \ldots a_n)(x, 1)^n = 0$; then, if

$$\frac{1}{\alpha_1 - x}, \quad \frac{1}{\alpha_2 - x}, \ldots \frac{1}{\alpha_n - x}$$

be substituted for $\alpha_1, \alpha_2, \ldots \alpha_n$, respectively, the result multiplied by U_n^{ϖ} (to remove fractions) is a *covariant* of U_n if it involves x, and an *invariant* if it does not involve x.

From this definition of an invariant we may infer at once that

$$a_0^{\varpi}\phi(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n)$$

is an invariant of U_n when ϕ is composed of a number of terms of the same type, each of which involves all the roots, and each root in the same degree ϖ.

These definitions may be extended to the case where ϕ (the function of differences) involves symmetrically the roots of several equations $U_p = 0$, $U_q = 0$, $U_r = 0$, &c., the roots of these equations entering ϕ in the orders ϖ, ϖ', ϖ'', &c. . . . respectively. We may substitute for each root α, $\dfrac{1}{\alpha - x}$ as before, and remove fractions by the multiplier $U_p^{\varpi} U_q^{\varpi'} U_r^{\varpi''}$ . . . . &c. If the result involves the variable x, we obtain a covariant of the system of quantics U_p, U_q, U_r, &c.; and if it does not, ϕ is an invariant of the system.

167. **Formation of Covariants and Invariants.**—We proceed now to show how the foregoing transformations may be conveniently effected, and covariants and invariants calculated in terms of the coefficients. With this object, let the seminvariant be expressed in terms of the coefficients as follows:—

$$a_0^{\varpi}\phi(\alpha_1, \alpha_2, \ldots \alpha_n) = F(a_0, a_1, a_2, \ldots a_n).$$

Now, changing the roots into their reciprocals, and consequently a_0 into a_n, &c., a_r into a_{n-r}, &c. (that is, giving the suffixes their complementary values), we have

$$a_0^{\varpi}\psi(\alpha_1, \alpha_2, \ldots \alpha_n) = F(a_n, a_{n-1}, \ldots a_0),$$

where ψ is an integral symmetric function of the roots, and F the corresponding value in terms of the coefficients. This function is called the *source** of the covariant derived therefrom.

Again, substituting $\alpha_1 - x$, $\alpha_2 - x$, . . . $\alpha_n - x$ for $\alpha_1, \alpha_2, \ldots \alpha_n$, and consequently U_r, &c., for a_r, &c. (Art. 35), we find

$$a_0^{\varpi}\psi(\alpha_1 - x, \alpha_2 - x, \ldots \alpha_n - x) = F(U_n, U_{n-1}, \ldots U_1, U_0).$$

Thus, by two steps we derive a covariant from a function of the differences, and find at the same time its equivalent calculated in terms of the coefficients.

To illustrate this mode of procedure, we take as an example, in the case of the cubic,

$$a_0^2\Sigma(\alpha - \beta)^2 = 18(a_1^2 - a_0 a_2);$$

* This term was introduced by Mr. Roberts.

whence, changing the roots into their reciprocals, and a_0, a_1, a_2, a_3 into a_3, a_2, a_1, a_0, we have

$$a_0^2\Sigma\alpha^2(\beta-\gamma)^2 = 18\,(a_2^2 - a_3a_1).$$

Again, changing α, β, γ into $\alpha - x, \beta - x, \gamma - x$, and a_1, a_2, a_3 into U_1, U_2, U_3, respectively, we find

$$a_0^2\Sigma\,(\beta-\gamma)^2(x-\alpha)^2 = 18\,(U_2^2 - U_3U_1).$$

The second member of this equation becomes when expanded

$$U_1U_3 - U_2^2 = (a_0a_2 - a_1^2)\,x^2 + (a_0a_3 - a_1a_2)\,x + (a_1a_3 - a_2^2).$$

This covariant is called the *Hessian* of U_3. We refer to it as H_x, since H is its leading coefficient.

As a second example we take the following function of the quartic:—

$$a_0^2\Sigma\,(\beta-\gamma)^2(\alpha-\delta)^2 = 24\,(a_0a_4 - 4a_1a_3 + 3a_2^2); \qquad (1)$$

whence, changing the roots into their reciprocals, and a_0, a_1, a_2, a_3, a_4 into a_4, a_3, a_2, a_1, a_0, we have

$$a_4^2\Sigma(\gamma-\beta)^2(\delta-\alpha)^2 = 24\,(a_4a_0 - 4a_3a_1 + 3a_2^2).$$

These transformations, therefore, do not alter equation (1): again, since in this case $\psi(\alpha, \beta, \gamma, \delta)$ is a function of the differences of the roots, ψ is unchanged when $\alpha - x$, $\beta - x$, &c. . . . , are substituted for $\alpha, \beta, \gamma, \delta$. We infer that $a_0a_4 - 4a_1a_3 + 3a_2^2$ is an invariant of the quartic U_4.

We observe also, in accordance with what was stated in Art. 166, since

$$\phi = (\beta-\gamma)^2(\alpha-\delta)^2 + (\gamma-\alpha)^2(\beta-\delta)^2 + (\alpha-\beta)^2(\gamma-\delta)^2,$$

that any one of the three terms of which ϕ is made up involves each of the roots in the degree ϖ, which is here equal to 2.

In a similar manner it may be shown that

$$a_0^3\{(\gamma-\alpha)(\beta-\delta) - (\alpha-\beta)(\gamma-\delta)\}\{(\alpha-\beta)(\gamma-\delta) - (\beta-\gamma)(\alpha-\delta)\}$$
$$\times\{(\beta-\gamma)(\alpha-\delta) - (\gamma-\alpha)(\beta-\delta)\}$$
$$= -432\,(a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 - a_2^3)$$

is an invariant of the quartic.

There is no difficulty in determining in any particular case whether ϕ leads to an invariant or covariant, for if it leads to an invariant, $\phi = \pm \psi$, that is, ϕ is unchanged (except in sign, when its type-term is the product of an odd number of differences of the roots, *i.e.* when its weight is odd) when for the roots their reciprocals are substituted, and fractions removed by the simplest multiplier $(a_1 a_2 a_3 \ldots a_n)^{\varpi}$. An invariant whose weight is odd is called a *skew invariant.*

168. **Properties of Covariants and Invariants.**—Since ϕ is a homogeneous function of the roots, the covariant derived from it may be written under the form

$$\frac{U^{\varpi}}{x^{\kappa}} \phi \left(\frac{x}{a_1 - x}, \quad \frac{x}{a_2 - x}, \ldots \frac{x}{a_n - x} \right),$$

where ϖ is the order, and κ the weight, of ϕ.

Also, since ϕ is a function of the differences, we may add 1 to each constituent, such as $\frac{x}{a_r - x}$, thus obtaining $\frac{a_r}{a_r - x}$. Again, multiplying each constituent by x, the covariant becomes

$$\frac{U^{\varpi}}{x^{2\kappa}} \phi \left(\frac{a_1 x}{a_1 - x}, \quad \frac{a_2 x}{a_2 - x}, \ldots \frac{a_n x}{a_n - x} \right).$$

Employing now the notation x', a'_1, a'_2, &c., for the reciprocals of x, a_1, a_2, &c.; and denoting by U' the function whose roots are $a'_1, a'_2, \ldots a'_n$, viz.

$$U' = a_n x'^n + n a_{n-1} x'^{n-1} + \text{\&c.}, \ldots + n a_1 x' + a_0 = 0;$$

since

$$\frac{1}{a'_r - x'} = \frac{-a_r x}{a_r - x},$$

and $\quad U = a_n x^n (x' - a'_1)(x' - a'_2) \ldots (x' - a'_n) = x^n U',$

the covariant above written is easily reduced to the form

$$(-1)^{\kappa} x^{n\varpi - 2\kappa} U'^{\varpi} \phi \left(\frac{1}{a'_1 - x'}, \quad \frac{1}{a'_2 - x'} \ldots \frac{1}{a'_n - x'} \right);$$

whence it is proved that the covariant is unaltered when for $x, a_1, a_2, \ldots a_n$ their reciprocals are substituted, and the result multiplied by $(-1)^\kappa x^{n\varpi-2\kappa}$. This transformation changes a_r into a_{n-r}, that is, each coefficient into the coefficient with the complementary suffix.

Now if any covariant whose degree is m be written in the form

$$(B_0, B_1, B_2, \ldots B_m)(x, 1)^m; \qquad (1)$$

changing $a_0, a_1, \ldots a_n, x$, into $a_n, a_{n-1}, \ldots a_0, \frac{1}{x}$, we have another form for this covariant, namely,

$$(-1)^\kappa x^{n\varpi-2\kappa}(C_0, C_1, C_2, \ldots C_m)\left(\frac{1}{x}, 1\right)^m;$$

and as this form is an integral function of x of the same type as (1), we have, by comparing the two forms,

$$m = n\varpi - 2\kappa, \quad B_0 = (-1)^\kappa C_m, \ldots B_r = (-1)^\kappa C_{m-r};$$

thus determining the degree of the covariant in terms of the order and weight of the function ϕ, and showing that the conjugate coefficients (*i.e.* those equally removed from the extremes) are related in the following way:—

If $F(a_0, a_1, a_2, \ldots a_n)$ *be any coefficient of the covariant,* $(-1)^\kappa F(a_n, a_{n-1}, a_{n-2}, \ldots a_0)$ *is its conjugate.*

This property is characteristic of covariants, and is not possessed by semicovariants, although the two classes of functions agree in the mode of formation by the operator D, as will appear in the Article which follows:—

From the expression for the degree of a covariant in terms of ϖ and κ, namely, $n\varpi - 2\kappa$, we may draw the following important inferences:—

(1). *If* $a_0{}^\varpi\phi$ *is an invariant,* $n\varpi = 2\kappa$.

For, in this case ϕ and ψ are the same function, and consequently their weights κ and $n\varpi - \kappa$ are also the same.

(2). *All the invariants of quantics of odd degrees are of even order.*

For if n be odd, it is plain from the equation $n\varpi = 2\kappa$ that ϖ must be even, and κ a multiple of n.

(3). *All covariants of quantics of even degrees are of even degrees.*

For in this case $n\varpi - 2\kappa$ is even.

(4). *Covariants of quantics of odd degrees are of odd or even degree according as the order of their coefficients is odd or even.*

(5). *The resultant of two covariants is always of an even order in the coefficients of the original quantic.*

For, the order of the resultant expressed in terms of the orders and weights of the covariants is

$$\varpi(n\varpi' - 2\kappa') + \varpi'(n\varpi - 2\kappa) = 2(n\varpi\varpi' - \varpi\kappa' - \varpi'\kappa).$$

169. **Formation of Covariants by the Operator D.**—From Art. 164 we infer that the expansion of $F(U_n, U_{n-1}, \ldots U_0)$ may be expressed by means of the Differential Calculus in the form

$$F_0 + xDF_0 + \frac{x^2}{1.2}D^2F_0 + \ldots + \frac{x^r}{1.2.3\ldots r}D^rF_0 + \ldots,$$

where F_0 is the result of making $x = 0$ in $F(U_n, U_{n-1}, \ldots U_0)$, viz.

$$F_0 = F(a_n, a_{n-1}, \ldots a_0),$$

and $$D \equiv a_0\frac{\partial}{\partial a_1} + 2a_1\frac{\partial}{\partial a_2} + 3a_2\frac{\partial}{\partial a_3} + \ldots + na_{n-1}\frac{\partial}{\partial a_n}.$$

In forming a covariant by this process, the source F_0 with which we set out is altered by the successive operations D, each operation reducing the weight by one, till we arrive at the original function $F(a_0, a_1, \ldots a_n)$ from which the source was formed. Since this is a function of the differences, the expression resulting from the next operation D vanishes, and the covariant is completely formed. The corresponding operations δ on the symmetric function ψ have the effect of reducing the degree in the roots by one each step, the final symmetric function containing the differences only. Thus by successive operations we obtain two expressions for a covariant—one in terms of the roots, and the other in terms of the coefficients.

The degree m of the covariant is plainly equal to the number of times δ operates in reducing ψ_0 to ϕ, *i.e.* equal to the difference of the weights of the extreme coefficients. And since

$$\psi_0 = (a_1 a_2 \ldots a_n)^{\varpi} \phi \left(\frac{1}{a_1}, \frac{1}{a_2}, \ldots \frac{1}{a_n}\right),$$

the weight of ψ_0 is $n\varpi - \kappa$, where κ is the weight of $\phi(a_1, a_2, \ldots a_n)$; hence the degree of the covariant whose leading coefficient is $a_0{}^{\varpi}\phi$ is $n\varpi - 2\kappa$, the same value as before obtained. We add some simple examples in illustration of this method.

Examples.

1. Form the Hessian of the cubic

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0.$$

Taking the function $H \equiv a_0a_2 - a_1{}^2$, we find, as in Art. 167,

$$a_0{}^2\Sigma a^2(\beta - \gamma)^2 = 18(a_2{}^2 - a_1a_3).$$

Operating on the left-hand side by δ, and on the right-hand side by D, we obtain

$$-a_0{}^2\Sigma 2a(\beta - \gamma)^2 = 18(a_1a_2 - a_0a_3);$$

and operating in the same way again,

$$a_0{}^2\Sigma 2(\beta - \gamma)^2 = 36(a_1{}^2 - a_0a_2).$$

The next operation causes both sides of the equation to vanish. Hence the required covariant is, as in Art. 167,

$$(a_1a_3 - a_2{}^2) + (a_0a_3 - a_1a_2)x + (a_0a_2 - a_1{}^2)x^2.$$

We find at the same time the corresponding expression in terms of x and the roots.

2. Form the Hessian of the biquadratic

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0.$$

The covariant whose leading coefficient is $H \equiv a_0a_2 - a_1{}^2$ is called the Hessian of the biquadratic. Its degree is 4, since $\varpi = 2$, and $\kappa = 2$; and $\therefore n\varpi - 2\kappa = 4$. Changing the coefficients into their complementaries, the source of the covariant is $a_4a_2 - a_3{}^2$, and we easily find

$$H_x \equiv (a_0a_2 - a_1{}^2)x^4 + 2(a_0a_3 - a_1a_2)x^3 + (a_0a_4 + 2a_1a_3 - 3a_2{}^2)x^2$$
$$+ 2(a_1a_4 - a_2a_3)x + (a_2a_4 - a_3{}^2).$$

3. Form for a cubic a covariant whose leading coefficient is the seminvariant G.

Changing the coefficients in G into their complementaries, we get the source $a_3^2a_0 - 3a_3a_2a_1 + 2a_2^3$, and operating with D we easily obtain the covariant in the following form:—

$$(a_3^2a_0 - 3a_3a_2a_1 + 2a_2^3) + 3(a_3a_2a_0 + a_2^2a_1 - 2a_3a_1^2)x$$
$$- 3(a_0a_1a_3 + a_1^2a_2 - 2a_0a_2^2)x^2 - (a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3)x^3.$$

In this the conjugate coefficients (Art. 168) differ in sign as well as in the interchange of complementaries, the weight of G being odd. The student will have no difficulty in expressing this covariant in terms of x and the roots by the aid of the value of G given in Ex. 15, Art. 27.

170. **Theorem.**—*Any function of the differences of the roots of a covariant or semicovariant is a function of the differences of the roots of the original equation.*

Let the covariant or semicovariant be

$$\phi(x) \equiv (x - \rho_1)(x - \rho_2) \ldots (x - \rho_p).$$

Since ϕ is a function of the differences of $x, a_1, a_2, \ldots a_n$, we have

$$\frac{\partial \phi}{\partial x} - \delta\phi = 0, \text{ viz., } \phi'(x) + \Sigma(x - \rho_2)(x - \rho_3) \ldots (x - \rho_p)\,\delta\rho_1 = 0.$$

Now, substituting for x in this identical equation each root $\rho_1, \rho_2, \ldots$ in succession, we have

$$\phi'(\rho_1)(1 + \delta\rho_1) = 0, \quad \phi'(\rho_2)(1 + \delta\rho_2) = 0, \&c., \ldots,$$

whence

$$\delta\rho_1 + 1 = 0, \quad \delta\rho_2 + 1 = 0, \ldots \delta\rho_j + 1 = 0, \ldots$$

and consequently

$$\delta(\rho_j - \rho_k) = 0,$$

which proves the theorem.

In the preceding pages many instances have been given in which the roots of covariants or semicovariants are expressed in terms of the roots of the original equation; and the student will easily verify that the result of the operation of δ on any such expression is -1. The roots of the covariants in Exs. 1 and 3 of the preceding Article are given in Ex. 25, p. 57, and Ex. 13, p. 88, Vol. I., respectively; and roots of semicovariants will be found in Exs. 10, 11, p. 87, and 12, 14, p. 88, Vol. I.

The theorem here proved is clearly true also for any function of the differences of the roots of two or more covariants or semicovariants.

171. **Double Linear Transformation applied to the Theory of Covariants.**—Hitherto we have discussed the theory of covariants and invariants through the medium of the roots of equations. We proceed now to give some account of a different and more general mode of treatment, by means of which this theory may be extended to quantics homogeneous in more than two variables, such as present themselves in the numerous important geometrical applications of the theory. Although this enlarged view of the subject does not come within the scope of the present work, we think it desirable to show the connexion between the method of treatment we have adopted and the more general method referred to. With this object we give in the present Article two important propositions.

PROP. I.—*Let any quantic*

$$U_n \equiv a_0 (x - \alpha_1 y)(x - \alpha_2 y) \ldots (x - \alpha_n y)$$

be transformed by the substitution

$$x = \lambda x' + \mu y', \quad y = \lambda' x' + \mu' y';$$

then if I and I' be corresponding invariants of the two forms U_n and U'_n, we have

$$I' = (\lambda\mu' - \lambda'\mu)^\kappa I.$$

To prove this, let

$$I = a_0^{\varpi} \Sigma (\alpha_1 - \alpha_2)^a (\alpha_2 - \alpha_3)^b \ldots (\alpha_1 - \alpha_n)^l,$$

each root entering every term of Σ in the degree ϖ. When any factor of U_n, e.g. $x - \alpha_1 y$, is transformed, we find

$$x - \alpha_1 y = (\lambda - \lambda'\alpha_1)(x' - \alpha'_1 y'), \text{ where } \alpha'_1 = \frac{\mu'\alpha_1 - \mu}{\lambda - \lambda'\alpha_1};$$

hence $\quad U'_n = a'_0 (x' - \alpha'_1 y')(x' - \alpha'_2 y') \ldots (x' - \alpha'_n y'),$

where $\quad a'_0 = a_0 (\lambda - \lambda'\alpha_1)(\lambda - \lambda'\alpha_2) \ldots (\lambda - \lambda'\alpha_n).$

Again, for the difference of any two roots of U'_n, we have

$$\alpha'_p - \alpha'_q = \frac{(\lambda\mu' - \lambda'\mu)(\alpha_p - \alpha_q)}{(\lambda - \lambda'\alpha_p)(\lambda - \lambda'\alpha_q)}.$$

Making these substitutions for a'_0 and for all the differences of roots in I', the denominators of the fractions which enter by the transformation disappear, and we have finally

$$I' = (\lambda\mu' - \lambda'\mu)^\kappa I.$$

Prop. II.—*If $\phi(x, y)$ be a covariant of the quantic U_n, the new value of ϕ, after linear transformation, is*

$$(\lambda\mu' - \lambda'\mu)^\kappa \phi(x, y).$$

The proof is similar to that of the preceding proposition. We have

$$\phi(x, y) = a_0^\varpi \Sigma (\alpha_1 - \alpha_2)^a (\alpha_2 - \alpha_3)^b \ldots (x - \alpha_1 y)^p (x - \alpha_2 y)^q \ldots,$$

where each root enters in the degree ϖ.

Now, transforming, as in the previous proposition, the value of $\phi(x, y)$ thus derived; since the factors $\lambda - \lambda'\alpha_1$, $\lambda - \lambda'\alpha_2$, ... all enter in the same degree ϖ in the denominator, they will all be removed by the multiplier $a_0'^\varpi$, and the transformed value of $\phi(x, y)$ is

$$(\lambda\mu' - \lambda'\mu)^\kappa \phi(x, y).$$

The determinant $\lambda\mu' - \lambda'\mu$, whose constituents are the coefficients which enter into the double linear transformation, is called the *modulus of transformation.*

Without any reference to the roots of the equation $U_n = 0$, we can suppose the transformation of x and y to be applied to the quantic in the form

$$U_n = a_0x^n + na_1x^{n-1}y + \frac{n(n-1)}{1 \cdot 2} a_2x^{n-2}y^2 + \ldots + a_ny^n.$$

The propositions here proved with respect to invariants and covariants regarded as functions of the roots will still hold good when these functions are expressed in their equivalent forms in terms of the coefficients. We may therefore now restate the Propositions in the following form:—

PROP. I.—*An invariant is a function of the coefficients of a quantic, such that when the quantic is transformed by linear transformation of the variables, the same function of the new coefficients is equal to the original function multiplied by a power of the modulus of transformation.*

PROP. II.—*A covariant is a function of the coefficients of a quantic, and also of the variables, such that when the quantic is transformed by linear transformation, the same function of the new variables and coefficients is equal to the original function transformed directly multiplied by a power of the modulus of transformation.*

The definitions contained in the preceding propositions are plainly applicable to quantics homogeneous in any number of variables, and form the basis of the more extended theory of covariants and invariants above referred to. We give among the following examples an application to the case of a quantic involving three variables.

EXAMPLES.

1. Performing the linear transformation

$$x = \lambda X + \mu Y, \quad y = \lambda_1 X + \mu_1 Y,$$

if

$$ax^2 + 2bxy + cy^2 = AX^2 + 2BXY + CY^2,$$

prove that

$$AC - B^2 = (\lambda\mu_1 - \lambda_1\mu)^2 (ac - b^2).$$

2. Performing the same transformation, if

$$(a, b, c, d, e)(x, y)^4 = (A, B, C, D, E)(X, Y)^4,$$

prove that

$$AE - 4BD + 3C^2 = (\lambda\mu_1 - \lambda_1\mu)^4 (ae - 4bd + 3c^2).$$

3. Performing the same transformation, if

$$ax^2 + 2bxy + cy^2 = AX^2 + 2BXY + CY^2,$$

and

$$a_1x^2 + 2b_1xy + c_1y^2 = A_1X^2 + 2B_1XY + C_1Y^2,$$

prove that

$$AC_1 + A_1C - 2BB_1 = (\lambda\mu_1 - \lambda_1\mu)^2 (ac_1 + a_1c - 2bb_1).$$

This follows from Ex. 1, applied to the quadratic forms

$$(a + \kappa a_1) x^2 + 2(b + \kappa b_1) xy + (c + \kappa c_1) y^2 = (A + \kappa A_1) X^2 + 2(B + \kappa B_1) XY + (C + \kappa C_1) Y^2,$$

by comparing the coefficients of κ on both sides.

Whence we may infer that, if two quadratics determine a harmonic system, the new quadratics obtained by linear transformation also form a harmonic system. For their roots being α, β, and α_1, β_1, we have

$$aa_1\{(\alpha-\alpha_1)(\beta-\beta_1)+(\alpha-\beta_1)(\beta-\alpha_1)\}=2\,(ac_1+a_1c-2bb_1).$$

4. If the homogeneous quadratic function of three variables

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$

be transformed into

$$AX^2+BY^2+CZ^2+2FYZ+2GZX+2HXY$$

by the linear substitution

$$x=\lambda_1X+\mu_1Y+\nu_1Z,\quad y=\lambda_2X+\mu_2Y+\nu_2Z,\quad z=\lambda_3X+\mu_3Y+\nu_3Z;$$

prove the relation

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = (\lambda_1\mu_2\nu_3)^2 \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix},$$

where the determinant $(\lambda_1\mu_2\nu_3)$ is the modulus of transformation.

This is easily verified by multiplying the proposed determinant of the original coefficients twice in succession by the modulus of transformation written in the form

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix}$$

and comparing the constituents of the resulting determinant with the expanded values of the coefficients of X^2, Y^2, &c., in the new form.

It appears therefore that the determinant here treated is an invariant of the given function of three variables.

172. **Properties of Covariants derived from Linear Transformation.**—We proceed now to show, taking the second statement of Prop. II. in Art. 171 as the definition of a covariant, that the law of derivation of the coefficients given in Art. 169 immediately follows:—that is, *given any one coefficient, all the rest may be determined.*

For this purpose, performing the linear transformation

$$x=X+hY,\quad y=0X+Y,$$

whose modulus is unity, the quantic

$(a_0, a_1, a_2, \ldots a_n)(x, y)^n$ becomes $(A_0, A_1, A_2, \ldots A_n)(X, Y)^n$,

where

$$A_0=a_0,\ \ A_1=a_1+a_0h,\ \ A_2=a_2+2a_1h+a_0h^2,\ \&c.\quad (\text{Art. } 35.)$$

Now, if $\phi(a_0, a_1, a_2, \ldots a_n, x, y)$ be any covariant of this quantic, we have by the definition

$$\phi(a_0, a_1, a_2, \ldots a_n, x, y) = \phi(A_0, A_1, A_2, \ldots A_n, X, Y),$$

or

$$\phi(a_0, a_1, a_2, \ldots a_n, x, y) = \phi(A_0, A_1, A_2, \ldots A_n, x - hy, y).$$

Expanding the second member of this equation, and confining our attention to the terms which multiply h: observing also that $\frac{\partial A_r}{\partial h} = ra_{r-1}$ when terms are omitted which would be multiplied in the result by h^2, h^3, &c., we have

$$\phi + h\left(-y\frac{\partial\phi}{\partial x} + D\phi\right) + h^2\left(\qquad \right) + \&c. \ldots \equiv \phi,$$

which must hold whatever value h may have; hence

$$y\frac{\partial\phi}{\partial x} = a_0\frac{\partial\phi}{\partial a_1} + 2a_1\frac{\partial\phi}{\partial a_2} + 3a_2\frac{\partial\phi}{\partial a_3} + \ldots + na_{n-1}\frac{\partial\phi}{\partial a_n}, \qquad (1)$$

and, substituting for ϕ the value

$$(B_0, B_1, B_2, \ldots B_m)(x, y)^m,$$

we have

$$mB_0x^{m-1}y + m(m-1)B_1x^{m-2}y^2 + \ldots + mB_{m-1}y^m$$

$$= DB_0x^m + mDB_1x^{m-1}y + \ldots + DB_my^m;$$

whence, comparing coefficients, we have the following equations :

$$DB_0 = 0, \quad DB_1 = B_0, \quad DB_2 = 2B_1, \ldots DB_m = mB_{m-1},$$

which determine the law of derivation of the coefficients from the source B_m; the leading coefficient B_0 being a function of the differences, since $DB_0 = 0$.

The calculation of the coefficients is facilitated by the following theorem, which has been proved already on different principles :—

Two coefficients of a covariant equally removed from the extremes become equal (plus or minus) when in either of them $a_0, a_1, \ldots a_n$ are replaced by $a_n, a_{n-1}, \ldots a_0$, respectively.

To prove this, let the quantic be transformed by the linea substitution

$$x = 0X + Y, \quad y = X + 0Y, \quad \text{whose modulus} = -1.$$

Thus

$$(a_0, a_1, a_2, \ldots a_n)(x, y)^n = (a_n, a_{n-1}, a_{n-2}, \ldots a_0)(X, Y)^n,$$

and, by definition, any covariant

$$\phi(a_n, a_{n-1}, a_{n-2}, \ldots a_0, X, Y) = (-1)^\kappa \phi(a_0, a_1, a_2, \ldots a_n, x, y)$$
$$= (-1)^\kappa \phi(a_0, a_1, a_2, \ldots a_n, Y, X);$$

whence it follows that the coefficients of the covariant equally removed from the extremes are similar in form, and become identical (except in sign when κ is odd) when for the suffixes their complementary values are substituted.

It is easily inferred in a similar manner that a covariant satisfies the differential equation

$$x\frac{\partial\phi}{\partial y} \equiv a_n\frac{\partial\phi}{\partial a_{n-1}} + 2a_{n-1}\frac{\partial\phi}{\partial a_{n-2}} + 3a_{n-2}\frac{\partial\phi}{\partial a_{n-3}} + \ldots + na_1\frac{\partial\phi}{\partial a_0}, \quad (2)$$

as well as the equation (1) already given.

Again, if $\phi(a_0, a_1, a_2, \ldots a_n)$ be an invariant of the quantic, the former transformation of the present Article gives, employing the definition of Art. 171,

$$\phi(a_0, a_1, a_2, \ldots a_n) = \phi(A_0, A_1, A_2, \ldots A_n);$$

and proceeding as before, in the case of a covariant, we prove that an invariant must satisfy both the differential equations

$$a_0\frac{\partial\phi}{\partial a_1} + 2a_1\frac{\partial\phi}{\partial a_2} + 3a_2\frac{\partial\phi}{\partial a_3} + \ldots + na_{n-1}\frac{\partial\phi}{\partial a_n} = 0,$$

$$a_n\frac{\partial\phi}{\partial a_{n-1}} + 2a_{n-1}\frac{\partial\phi}{\partial a_{n-2}} + 3a_{n-2}\frac{\partial\phi}{\partial a_{n-3}} + \ldots + na_1\frac{\partial\phi}{\partial a_0} = 0,$$

either of which may be regarded as contained in the other, since if we make the linear transformation $x = Y$, $y = X$ (whose modulus $= -1$), we have from the definition of an invariant

$$\phi(a_n, a_{n-1}, a_{n-2}, \ldots a_0) = (-1)^\kappa\phi(a_0, a_1, a_2, \ldots a_n);$$

proving that an invariant is a function of the coefficients of a quantic which does not alter (except in sign if the weight be odd) when the coefficients are written in direct or reverse order.

The relation between invariants and seminvariants, covariants and semicovariants, is now clear. Invariants of the quantic $(a_0, a_1, \ldots a_n)(x, y)^n$ satisfy both the differential equations last written, whereas seminvariants of $(a_0, a_1, \ldots a_n)(x, 1)^n$ satisfy only the first of these equations. In like manner semicovariants of $(a_0, a_1, \ldots a_n)(x, 1)^n$ satisfy only the first of the differential equations (1) and (2) above written, whereas both are satisfied by covariants.

Having now explained the nature of Covariants and Invariants of quantics, and the connexion between the two modes in which these functions may be discussed, we proceed to prove certain propositions which are of wide application in the formation of the Covariants and Invariants of quantics transformed by a linear substitution. The student who is reading this subject for the first time may pass at once to the next chapter, where the principles already explained are applied to the cases of the quadratic, cubic, and quartic.

173. Prop. I.—*Let any homogeneous quantic of the n^{th} degree $f(x, y)$ become $F(X, Y)$ by the linear transformation*

$$x = \lambda X + \mu Y, \quad y = \lambda' X + \mu' Y;$$

also let any function u, of x, y, become U by the same transformation; then we have

$$M^n f\left(\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x}\right) = F\left(\frac{\partial U}{\partial Y}, -\frac{\partial U}{\partial X}\right), \qquad (1)$$

where M is the Modulus of transformation.

To prove this proposition, solving the equations

$$x = \lambda X + \mu Y, \quad y = \lambda' X + \mu' Y,$$

we have

$$MX = \mu' x - \mu y, \quad MY = -\lambda' x + \lambda y;$$

whence

$$M\frac{\partial X}{\partial x} = \mu', \quad M\frac{\partial X}{\partial y} = -\mu, \quad M\frac{\partial Y}{\partial x} = -\lambda', \quad M\frac{\partial Y}{\partial y} = \lambda.$$

Again, $$\frac{\partial u}{\partial x} = \frac{\partial U}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial U}{\partial Y}\frac{\partial Y}{\partial x} = \frac{1}{M}\left(\mu'\frac{\partial U}{\partial X} - \lambda'\frac{\partial U}{\partial Y}\right),$$

$$\frac{\partial u}{\partial y} = \frac{\partial U}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial U}{\partial Y}\frac{\partial Y}{\partial y} = \frac{1}{M}\left(-\mu\frac{\partial U}{\partial X} + \lambda\frac{\partial U}{\partial Y}\right),$$

which equations may be put under the form

$$\frac{\partial u}{\partial y} = \lambda\left(\frac{1}{M}\frac{\partial U}{\partial Y}\right) + \mu\left(-\frac{1}{M}\frac{\partial U}{\partial X}\right),$$

$$-\frac{\partial u}{\partial x} = \lambda'\left(\frac{1}{M}\frac{\partial U}{\partial Y}\right) + \mu'\left(-\frac{1}{M}\frac{\partial U}{\partial X}\right);$$

and since

$$f(\lambda X + \mu Y, \quad \lambda' X + \mu' Y) \equiv F(X, Y),$$

changing X and Y into $\frac{1}{M}\frac{\partial U}{\partial Y}$, and $-\frac{1}{M}\frac{\partial U}{\partial X}$, respectively, the proposition is proved.

In an exactly similar manner, changing X and Y into

$$\frac{1}{M}\frac{\partial}{\partial Y}, \quad -\frac{1}{M}\frac{\partial}{\partial X},$$

it may be proved that

$$M^n f\left(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}\right)u = F\left(\frac{\partial}{\partial Y}, -\frac{\partial}{\partial X}\right)U. \qquad (2)$$

The results (1) and (2) may be applied to generate covariants and invariants, as we proceed to show.

Suppose $f(x, y)$ and u to be covariants of any third quantic v, where v may become identical with either as a particular case; also, denoting by $F_c(X, Y)$ and U_c the same covariants expressed in terms of the X, Y variables and the new coefficients of v after linear transformation, we have, by Prop. II., Art. 171, the identical equations

$$M^p F(X, Y) = F_c(X, Y), \text{ and } M^q U = U_c;$$

whence, substituting from these equations in (1),

$$M^r f\left(\frac{\partial u}{\partial y}, \ -\frac{\partial u}{\partial x}\right) = F_c\left(\frac{\partial U_c}{\partial Y}, \ -\frac{\partial U_c}{\partial X}\right),$$

proving that $f\left(\frac{\partial u}{\partial y}, \ -\frac{\partial u}{\partial x}\right)$ is a covariant of v.

And in a similar manner it is proved from (2) that

$$f\left(\frac{\partial}{\partial y}, \ -\frac{\partial}{\partial x}\right)u$$

leads to an invariant or covariant of v, according as u is of the n^{th} or any higher order.

We add some applications of this method of forming invariants and covariants.

Examples.

1. If $\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}$ be substituted for x and y in the quartic $(a, b, c, d, e)(x, y)^4 \equiv U$, and the resulting operation performed on the quartic itself, show that the invariant I is obtained.

We find

$$(a, b, c, d, e)\left(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}\right)^4 U = 48(ae - 4bd + 3c^2).$$

2. Prove, by performing the same operation on H_x, the Hessian of the quartic (Ex. 2, Art. 169), that the invariant J is obtained.

Here we find

$$(a, b, c, d, e)\left(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}\right)^4 H_x = 72(ace + 2bcd - ad^2 - eb^2 - c^3).$$

3. Prove that

$$(a, b, c, d)\left(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x}\right)^3 G_x = -12(a^2d^2 - 6abcd + 4ac^3 + 4b^3d - 3b^2c^2),$$

where G_x is the cubic covariant of the cubic $(a, b, c, d)(x, y)^3$ (Ex. 3, Art. 169).

4. Find the value of

$$(ac - b^2)\left(\frac{\partial u}{\partial y}\right)^2 - (ad - bc)\frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + (bd - c^2)\left(\frac{\partial u}{\partial x}\right)^2,$$

where $u \equiv (a, b, c, d)(x, y)^3$.

Ans. $-9H_x^2$.

174. Prop. II.—*If* $\phi\,(a_0, a_1, a_2, \ldots a_n)$ *be an invariant of the form* $(a_0, a_1, a_2, \ldots a_n)\,(x, y)^n$, *and* u *any quantic of the* n^{th} *or any higher degree,*

$$\phi\left(\frac{\partial^n u}{\partial x^n}, \frac{\partial^n u}{\partial x^{n-1}\partial y}, \frac{\partial^n u}{\partial x^{n-2}\partial y^2}, \ldots \frac{\partial^n u}{\partial y^n}\right)$$

is an invariant or covariant of u.

To prove this, let

$$x = \lambda X + \mu Y, \quad x' = \lambda X' + \mu Y',$$
$$y = \lambda' X + \mu' Y, \quad y' = \lambda' X' + \mu' Y';$$

and transforming, as in the last proposition,

$$x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} = X'\frac{\partial}{\partial X} + Y'\frac{\partial}{\partial Y};$$

also transforming u, we have $U = u$; whence

$$\left(X'\frac{\partial}{\partial X} + Y'\frac{\partial}{\partial Y}\right)^n U = \left(x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}\right)^n u; \qquad (1)$$

and writing this equation when expanded under the form

$$(D_0, D_1, D_2, \ldots D_n)\,(X', Y')^n = (d_0, d_1, d_2, \ldots d_n)\,(x', y')^n,$$

we have, from the definition of an invariant,

$$\phi(D_0, D_1, D_2, \ldots D_n) = M^q\phi(d_0, d_1, d_2, \ldots d_n),$$

showing that $\phi(d_0, d_1, d_2, \ldots d_n)$ is an invariant or covariant.

When x, y, and x', y' are transformed similarly, as in the present proposition, they are said to be *cogredient* variables. And, in general, for any number of variables, when the coefficients which enter into the transformation of one set are the same as those which enter into the transformation of the other, the two sets are said to be cogredient.

The functions which occur in the equation (1) are called *emanants*; the expression on the right-hand side of the equation being the n^{th} emanant of u.

EXAMPLES.

1. Let the quadratic

$$a_0x^2 + 2a_1xy + a_2y^2 \quad \text{become} \quad A_0X^2 + 2A_1XY + A_2Y^2.$$

We have then, as in Ex. 1, Art. 171,

$$A_0A_2 - A_1^2 = M^2(a_0a_2 - a_1^2).$$

Now since

$$X'^2\frac{\partial^2 U}{\partial X^2} + 2X'Y'\frac{\partial^2 U}{\partial X\partial Y} + Y'^2\frac{\partial^2 U}{\partial Y^2} = x'^2\frac{\partial^2 u}{\partial x^2} + 2x'y'\frac{\partial^2 u}{\partial x\partial y} + y'^2\frac{\partial^2 u}{\partial y^2}$$

it follows from the last result, considering X', Y', and x', y' as variables, that

$$\frac{\partial^2 U}{\partial X^2}\frac{\partial^2 U}{\partial Y^2} - \left(\frac{\partial^2 U}{\partial X\partial Y}\right)^2 = M^2\left\{\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x\partial y}\right)^2\right\}$$

This gives an invariant of a quadratic, and a covariant (called the *Hessian*) of any higher quantic.

2. When u has the values

$$(a, b, c, d)(x, y)^3 \quad \text{and} \quad (a, b, c, d, e)(x, y)^4,$$

what covariants are derived by the process of the last example?

(Cf. Exs. 1, 2, Art. 169.)

Ans. (1). $(ac - b^2)x^2 + (ad - bc)xy + (bd - c^2)y^2$.

(2). $(ac - b^2)x^4 + 2(ad - bc)x^3y + (ae + 2bd - 3c^2)x^2y^2 + 2(be - cd)xy^3 + (ce - d^2)y^4$.

175. PROP. III.—*If any invariant of the quantic in x, y,*

$$U + k(xy' - x'y)^n$$

be formed, the coefficients of the different powers of k, regarded as homogeneous functions of the variables x', y', are covariants of U.

For, transforming U by linear transformation, let

$$(a_0, a_1, a_2, \ldots a_n)(x, y)^n = (A_0, A_1, A_2, \ldots A_n)(X, Y)^n;$$

also, if x, y and x', y' be cogredient variables,

$$xy' - x'y = M(XY' - X'Y).$$

Whence

$$(a_0, a_1, a_2, \ldots a_n)(x, y)^n + k(xy' - x'y)^n$$

becomes when transformed

$$(A_0, A_1, A_2, \ldots A_n)(X, Y)^n + kM^n(XY' - X'Y)^n;$$

and forming any invariant ϕ of both these forms, we have

$$(\phi, \phi_1, \phi_2, \ldots \phi_p)(1, k)^p = M^\kappa(\Phi, \Phi_1 \Phi_2, \ldots \Phi_p)(1, M^n k)^p,$$

proving that

$$\phi_r = M^q \Phi_r,$$

or that ϕ_r is a covariant.

When $(xy' - x'y)^n$ is replaced by $(b_0, b_1, b_2, \ldots b_n)(x, y)^n$, we have the following proposition, which is established in a similar manner :—

If $\phi(a_0, a_1, a_2, \ldots a_n)$ be an invariant of $(a_0, a_1, a_2, \ldots a_n)$ $(x, y)^n$ all the coefficients of k in

$$\phi(a_0 + kb_0, \quad a_1 + kb_1, \ldots a_n + kb_n)$$

are invariants of the system of two quantics

$$(a_0, a_1, a_2, \ldots a_n)(x, y)^n, \quad (b_0, b_1, b_2, \ldots b_n)(x, y)^n;$$

or, which is the same thing,

$$\left(b_0 \frac{\partial}{\partial a_0} + b_1 \frac{\partial}{\partial a_1} + \ldots + b_n \frac{\partial}{\partial a_n}\right)^r \phi, \text{ \&c., \&c.,}$$

are invariants of the system.

This proposition may be extended to any number of quantics of the same degree in any number of variables. If, further, U be replaced by a covariant V of the p^{th} degree, we may generate new covariants by forming any invariant of

$$V + k(xy' - x'y)^p.$$

176. Prop. IV.—*If $\phi(x, y)$ and $\psi(x, y)$ be homogeneous quantics, the determinant*

$$\begin{vmatrix} \dfrac{\partial\phi}{\partial x} & \dfrac{\partial\phi}{\partial y} \\ \\ \dfrac{\partial\psi}{\partial x} & \dfrac{\partial\psi}{\partial y} \end{vmatrix}$$

is a covariant of these quantics.

For, transforming ϕ and ψ by the linear substitution

$$x = \lambda X + \mu Y, \quad y = \lambda' X + \mu' Y,$$

we have

$$\Phi(X, Y) = \phi(x, y), \qquad \Psi(X, Y) = \psi(x, y),$$

giving

$$\frac{\partial \Phi}{\partial X} = \lambda \frac{\partial \phi}{\partial x} + \lambda' \frac{\partial \phi}{\partial y}, \quad \frac{\partial \Psi}{\partial X} = \lambda \frac{\partial \psi}{\partial x} + \lambda' \frac{\partial \psi}{\partial y},$$

$$\frac{\partial \Phi}{\partial Y} = \mu \frac{\partial \phi}{\partial x} + \mu' \frac{\partial \phi}{\partial y}, \quad \frac{\partial \Psi}{\partial Y} = \mu \frac{\partial \psi}{\partial x} + \mu' \frac{\partial \psi}{\partial y}.$$

Whence

$$\begin{vmatrix} \dfrac{\partial \Phi}{\partial X} & \dfrac{\partial \Phi}{\partial Y} \\ \dfrac{\partial \Psi}{\partial X} & \dfrac{\partial \Psi}{\partial Y} \end{vmatrix} = \begin{vmatrix} \lambda \dfrac{\partial \phi}{\partial x} + \lambda' \dfrac{\partial \phi}{\partial y}, & \mu \dfrac{\partial \phi}{\partial x} + \mu' \dfrac{\partial \phi}{\partial y} \\ \lambda \dfrac{\partial \psi}{\partial x} + \lambda' \dfrac{\partial \psi}{\partial y}, & \mu \dfrac{\partial \psi}{\partial x} + \mu' \dfrac{\partial \psi}{\partial y} \end{vmatrix},$$

which reduces to

$$M\left(\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial x}\right);$$

and the proposition is proved.

This covariant is called the *Jacobian* of ϕ and ψ, and is often written in the form $J(\phi, \psi)$. The Jacobian of n functions in n variables is a determinant of similar form, and can be shown to be a covariant by an exactly similar proof.

177. **Derivation of Invariants and Covariants by Differential Symbols.**—If x_1, y_1; x_2, y_2; x_3, y_3; ... x_n, y_n be a series of cogredient variables (such as, for example, the coordinates of n points), the functions $(x_1 y_2 - x_2 y_1)$, ... $(x_p y_q - x_q y_p)$ are unaltered by linear transformation; and since $\dfrac{\partial}{\partial y_i}, -\dfrac{\partial}{\partial x_i}$ are transformed by the same linear transformation as x_i, y_i (see Art. 173), we derive a series of symbols of differentiation, which, combined as above, give the following :—

$$\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial y_2} - \frac{\partial}{\partial x_2}\frac{\partial}{\partial y_1}\right), \; \ldots \; \left(\frac{\partial}{\partial x_p}\frac{\partial}{\partial y_q} - \frac{\partial}{\partial x_q}\frac{\partial}{\partial y_p}\right), \text{ \&c.}$$

These symbols may be denoted simply by (1, 2), . . . (p, q), &c.; and by their aid a complete calculus can be constructed for deriving and comparing invariants and covariants. For example, the Jacobian of ϕ, ψ may be written in the form

$$(1, 2)\,\phi_1\,\psi_2,$$

where $\phi_1 = \phi(x_1, y_1), \quad \psi_2 = \psi(x_2, y_2),$

the suffixes being omitted after the differentiation has been performed. Similarly, expanding the symbolic form $(1, 2)^2\phi_1\psi_2$, we obtain the covariant

$$\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\psi}{\partial y^2} - 2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2\psi}{\partial x\partial y} + \frac{\partial^2\phi}{\partial y^2}\frac{\partial^2\psi}{\partial x^2},$$

the distinction between the variables being removed after the differentiation has been performed.

In the investigation by this method of the invariants and covariants of a single quantic, the result is obtained under the symbolic form

$$(1, 2)^\alpha (2, 3)^\beta (3, 4)^\gamma \ldots (p, q)^\lambda U_1 U_2 U_3 \ldots U_p U_q,$$

where U_j, for example, is used to denote the quantic obtained by substituting x_j and y_j for x and y in U. If after this operation is performed, x and y disappear, we have obtained an invariant; and it is easy to see in this case that the figures 1, 2, 3, . . . p, q must all occur exactly n times in terms such as $(i, j)^\kappa$. For example, the formula

$$(1, 2)^n U_1 U_2$$

gives a series of binary invariants for all *even* quantics, the order of the invariant in general being equal to the number of factors U_1, U_2, &c. In like manner from the formula

$$(12)^{2m}(23)^{2m}(31)^{2m}\,U_1U_2U_3$$

we can derive a series of ternary invariants for quantics of the degree $4m$, the operation $(12)^2(23)^2(31)^2$ in the case of the quartic yielding the invariant

$$a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_4a_1^2 - a_2^3.$$

It may be noticed that this interchange of variables can be accomplished by means of a differential operator; for instance,

$$\left(x_\kappa \frac{\partial}{\partial x_j} + y_\kappa \frac{\partial}{\partial y_j}\right)^n U_j = 1 \,.\, 2 \,.\, 3 \ldots n U_\kappa, \text{ \&c., \&c.}$$

The method here explained of forming invariants and covariants is due to Prof. Cayley.

The above method of calculating invariants and covariants can be easily extended to ternary forms; for, if $x_1y_1z_1$, $x_2y_2z_2$, $x_3y_3z_3$ be cogredient variables, it appears readily by the rule for multiplying determinants that if we express $\partial/\partial X_1$, $\partial/\partial Y_1$, $\partial/\partial Z_1$, in terms of $\partial/\partial x_1$, $\partial/\partial y_1$, $\partial/\partial z_1$, and deal similarly with the other partial differential coefficients, the following relations hold between symbols of differentiation :—

$$\begin{vmatrix} \frac{\partial}{\partial X_1} & \frac{\partial}{\partial Y_1} & \frac{\partial}{\partial Z_1} \\ \frac{\partial}{\partial X_2} & \frac{\partial}{\partial Y_2} & \frac{\partial}{\partial Z_2} \\ \frac{\partial}{\partial X_3} & \frac{\partial}{\partial Y_3} & \frac{\partial}{\partial Z_3} \end{vmatrix} = M \begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial y_1} & \frac{\partial}{\partial z_1} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial y_2} & \frac{\partial}{\partial z_2} \\ \frac{\partial}{\partial x_3} & \frac{\partial}{\partial y_3} & \frac{\partial}{\partial z_3} \end{vmatrix} \equiv M\,(123),$$

where M is the modulus of the transformation.

178. **Notation of Aronhold and Clebsch.**—Aronhold and Clebsch have used with much success a method of forming invariants and covariants which is closely allied to the method given by Cayley. It is therefore desirable to explain their notation, and to show the connexion of the two methods of procedure.

Aronhold denotes symbolically the binary quantic U of the n^{th} degree by $a_x^n \equiv (a_1x_1 + a_2x_2)^n$. The products $a_1^p a_2^q$ are at once expressible by the coefficients of U when $p + q = n$. Thus, a_1^n denotes the coefficient of x_1^n in U, $a_1^{n-1}a_2$ the coefficient of

$x_1^{n-1}x_2$, and so on; but when the sum of the indices $p+q$ in the product $a_1^p a_2^q$ is not a multiple of n, no interpretation is afforded.

Again, since

$$U = \frac{1}{1\,.\,2\,.\,3\,\ldots\,.\,n}\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}\right)^n U = (a_1x_1 + a_2x_2)^n,$$

we may replace a_1 and a_2 in any homogeneous function of a_1 and a_2 of the n^{th} degree by the differential symbols $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_2}$ operating on U, a numerical factor being disregarded.

Moreover, in place of substituting different pairs of variables in U, thus forming $U_1, U_2, U_3, \ldots$, as in Cayley's method, Aronhold writes the quantic U under the various forms $(a_1x_1 + a_2x_2)^n$, $(b_1x_1 + b_2x_2)^n$, $(c_1x_1 + c_2x_2)^n$, . . ., where $a_1^p a_2^{n-p} = b_1^p b_2^{n-p} = c_1^p c_2^{n-p} \ldots$, the coefficient of $x_1^p x_2^{n-p}$ in U, the order of the coefficients of any invariant or covariant being the number of the symbols a, b, c, in its expression. In the formation of an invariant the differential symbol $\left(\frac{\partial}{\partial x_1}\frac{\partial}{\partial y_2} - \frac{\partial}{\partial x_2}\frac{\partial}{\partial y_1}\right)$ given in Cayley's method is now replaced by $(ab) = (a_1b_2 - a_2b_1)$ in Aronhold's notation.

Thus, for example, the invariants of the quartic are, in Aronhold's notation, written thus :—

$$2I = (ab)^4, \quad 6J = (bc)^2\,(ca)^2\,(ab)^2,$$

and the covariants whose leading coefficients are H and G as follows :—

$$2H_x = (ab)^2 a_x^2 b_x^2, \quad G_x = (ab)^2\,(ca)\,a_x b_x^2 c_x^3,$$

which expressions may be verified by replacing (ab), &c., by $(a_1b_2 - a_2b_1)$, &c., then expanding and introducing the coefficients of U, which is practicable since these expressions are homogeneous functions of the 4^{th} degree in each pair of symbols a_1, a_2. This method, like Cayley's, can be readily applied to a quantic U involving any number of variables.

We now conclude this chapter with some examples selected to illustrate the foregoing theory. The student is referred for further information on this subject to Salmon's *Lessons Introductory to the Modern Higher Algebra*; to Gordan's *Vorlesungen über Invariantentheorie*; and to Clebsch's *Theorie der binären algebraischen Formen*, where a symbolic method is adopted throughout.

Examples.

1. The discriminant of any quantic is an invariant.

2. The resultant of two quantics is an invariant of the system.

3. From the definitions, Art. 166, prove that all the invariants of the quantic $(xy' - x'y)U$ are covariants of U, the variable being $x' : y'$.

Hence derive the covariants of a cubic from the invariants of a quartic expressed in terms of the roots.

4. If $I_1, I_2, I_3, \ldots I_n$ be the same invariant for each of the quantics

$$\frac{\phi(x)}{x - a_1}, \ \frac{\phi(x)}{x - a_2}, \ \frac{\phi(x)}{x - a_3}, \ldots \frac{\phi(x)}{x - a_n},$$

of the order ϖ, where $a_1, a_2, \ldots a_n$ are the roots of $\phi(x) = 0$, prove that

$$\sum_{r=1}^{r=n} I_r(x - a_r)^{\varpi}$$

is a covariant of $\phi(x)$.

For example, using J_1 to denote the J invariant composed of the four roots a_2, a_3, a_4, a_5 (Art. 167), with similar values for J_2, J_3, J_4, J_5, we have the following covariant of a quintic :—

$$J_1(x - a_1)^3 + J_2(x - a_2)^3 + J_3(x - a_3)^3 + J_4(x - a_4)^3 + J_5(x - a_5)^3.$$

5. If $a_1, a_2, a_3, \ldots a_n$ be the roots of the equation

$$(a_0, a_1, a_2, \ldots a_n)(x, 1)^n = 0;$$

and if

$$a_0{}^{\varpi}\phi_1\phi_2 \ldots \phi_m = F(a_0, a_1, a_2, \ldots a_n),$$

where $\phi_1, \phi_2, \ldots \phi_m$ are all the values of a rational and integral function of some or all the roots obtained by substitution, find the equation whose roots are the m values of $-\dfrac{\phi}{\delta\phi}$, given $\delta^2\phi = 0$. (Cf. Exs. 12, 13, 14, p. 88, Vol. I.)

Ans. $F(U_0, U_1, U_2, \ldots U_n) = 0$.

6. Express the identical relation connecting three quadratics in terms of their invariants.

Let
$$U = a_1x^2 + 2b_1xy + c_1y^2,$$
$$V = a_2x^2 + 2b_2xy + c_2y^2,$$
$$W = a_3x^2 + 2b_3xy + c_3y^2;$$

multiplying together the two determinants

$$\begin{vmatrix} a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ a_3 & b_3 & c_3 & 0 \\ y^2 & -xy & x^2 & 0 \end{vmatrix} \quad \begin{vmatrix} c_1 & -2b_1 & a_1 & 0 \\ c_2 & -2b_2 & a_2 & 0 \\ c_3 & -2b_3 & a_3 & 0 \\ x^2 & 2xy & y^2 & 0 \end{vmatrix}$$

we have

$$4\begin{vmatrix} I_{11} & I_{12} & I_{13} & U \\ I_{12} & I_{22} & I_{23} & V \\ I_{13} & I_{23} & I_{33} & W \\ U & V & W & 0 \end{vmatrix} \equiv 0, \text{ where } 2I_{pq} = a_pc_q + a_qc_p - 2b_pb_q.$$

Expanding this determinant we have

$$(I_{22}I_{33} - I_{23}^2)U^2 + (I_{33}I_{11} - I_{31}^2)V^2 + (I_{11}I_{22} - I_{12}^2)W^2 + 2(I_{31}I_{12} - I_{11}I_{23})VW$$
$$+ 2(I_{23}I_{12} - I_{22}I_{31})WU + 2(I_{23}I_{31} - I_{33}I_{12})UV \equiv 0. \qquad (1)$$

There are two particular cases worth noticing :—

(1). *When the three quadratics are mutually harmonic.*—In this case $I_{23} = 0$, $I_{31} = 0$, $I_{12} = 0$; and the identical equation assumes the following simple form :

$$\left(\frac{U}{\sqrt{I_{11}}}\right)^2 + \left(\frac{V}{\sqrt{I_{22}}}\right)^2 + \left(\frac{W}{\sqrt{I_{33}}}\right)^2 \equiv 0.$$

(2). *When one of the quadratics $W = 0$ determines the foci of the involution of the points given by the other two, $U = 0$, and $V = 0$.*—In this case $I_{13} = 0$, and $I_{23} = 0$; and making this reduction in the general equation (1), we have

$$(I_{12}^2 - I_{11}I_{22})W^2 = I_{33}(I_{22}U^2 - 2I_{12}UV + I_{11}V^2);$$

but from the equations $I_{13} = 0$, and $I_{23} = 0$, we find

$$a_3 = \kappa(a_1b_2), \quad -2b_3 = \kappa(c_1a_2), \quad c_3 = \kappa(b_1c_2);$$

whence

$$4(a_3c_3 - b_3^2) = \kappa^2\{4(a_1b_2)(b_1c_2) - (c_1a_2)^2\},$$

or

$$I_{33} = \kappa^2\{I_{11}I_{22} - I_{12}^2\},$$

and reducing, when $\kappa = 1$, or $W \equiv J(U, V)$,

$$-\{J(U, V)\}^2 = I_{22}U^2 - 2I_{12}UV + I_{11}V^2.$$

7. Prove that

$$\Sigma\nabla(a_2, a_3, a_4)(x - a_1)^4$$

is a covariant of a quartic, where $\nabla(a_1, a_2, \ldots a_r)$ represents the product of the squared differences of $a_1, a_2, \ldots a_r$.

8. Prove that the condition that four roots of an equation of the n^{th} degree should determine on a right line a harmonic system of points may be expressed by equating to zero an invariant of the degree $\frac{1}{2}(n-1)(n-2)(n-3)$.

9. If $\phi(a_0, a_1, \ldots a_n)$ be any seminvariant of the quantic $(a_0, a_1, \ldots a_n)(x, 1)^n$, prove that $\dfrac{\partial\phi}{\partial a_n}$ is also a seminvariant.

10. Prove that the seminvariants

$$a_0a_2 - a_1^2, \quad a_0a_4 - 4a_1a_3 + 3a_2^2, \quad a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3,$$

of the quantic $(a_0, a_1, a_2, \ldots a_n)(x, y)^n$ give rise to covariants of the degrees

$$2n-4, \qquad 2n-8, \qquad 3n-6.$$

11. Prove that the coefficient of the penultimate term in the equation of the squares of the differences of any quantic leads to a covariant of that quantic of the fourth degree in the variables.

12. Prove that the product of two covariants of the same quantic whose sources are ϕ and ψ may be written under the form

$$\phi\psi + xD(\phi\psi) + \frac{x^2}{1 \cdot 2}D^2(\phi\psi) + \&c. \ldots$$

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(See Art. 160.)

13. Prove in particular that the m^{th} power of the quantic

$$(a_0, a_1, a_2, \ldots a_n)(x, 1)^n$$

may be represented by

$$a_n^m + xD(a_n^m) + \frac{x^2}{1 \cdot 2}D^2(a_n^m) + \frac{x^3}{1 \cdot 2 \cdot 3}D^3(a_n^m) + \&c.$$

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14. Prove from both definitions of a covariant that any covariant of a covariant is a covariant of the original quantic or quantics.

15. If $a_1, a_2, \ldots a_m$, and $\beta_1, \beta_2, \ldots \beta_n$ be the roots of the equations $U \equiv (a_0, a_1, a_2, \ldots a_m)(x, 1)^m = 0$, and $V \equiv (b_0, b_1, b_2, \ldots b_n)(x, 1)^n = 0$; it is required to derive a covariant of the system U and V from the simplest function of the differences of their roots, viz., $\Sigma(a_p - \beta_q) \equiv n\Sigma a - m\Sigma\beta$.

This question will be solved if we express

$$UV\sum\frac{a_p - \beta_q}{(x - a_p)(x - \beta_q)}$$

in terms of the coefficients of U and V.

For this purpose we have

$$\sum\frac{a_p - \beta_q}{(x - a_p)(x - \beta_q)} \equiv \sum\frac{a}{x - a}\sum\frac{1}{x - \beta} - \sum\frac{\beta}{x - \beta}\sum\frac{1}{x - a};$$

and if U and V be written as homogeneous functions of x and y,

$$\sum\frac{1}{x - ay} = \frac{\partial \log U}{\partial x}, \quad \sum\frac{a}{x - ay} = -\frac{\partial \log U}{\partial y}, \&c.$$

Whence, substituting these values in the last equation, we have

$$UV\sum\frac{\alpha_p-\beta_q}{(x-\alpha_p y)(x-\beta_q y)}=\frac{\partial U}{\partial x}\frac{\partial V}{\partial y}-\frac{\partial U}{\partial y}\frac{\partial V}{\partial x};$$

which is the Jacobian of U and V. It should be noticed also that the leading coefficient of $J(U, V)$ is $mn(a_0b_1-a_1b_0)$.

16. Prove that the common factors of two quantics are double factors of their Jacobian $J(U, V)$, when the quantics are of the same degree n.

Let $U \equiv P\phi$, $V \equiv P\psi$, where $P \equiv lx + my$. Forming $J(U, V)$, we find part of it divisible by P^2, and the part which apparently has only P as a factor may be written as follows (using Euler's theorem of homogeneous functions, and omitting a numerical factor):—

$$\left(x\frac{\partial\phi}{\partial x}+y\frac{\partial\phi}{\partial y}\right)\left(l\frac{\partial\psi}{\partial y}-m\frac{\partial\psi}{\partial x}\right)+\left(x\frac{\partial\psi}{\partial x}+y\frac{\partial\psi}{\partial y}\right)\left(m\frac{\partial\phi}{\partial x}-l\frac{\partial\phi}{\partial y}\right),$$

and this is identical with $(lx + my)J(\phi, \psi)$.

17. Prove that the $2(n-1)$ double factors of $\lambda U + \mu V$, obtained by varying λ and μ, are the factors of $J(U, V)$, where U and V are both of the n^{th} degree.

18. Find the resultant of two cubics U and V by eliminating dialytically between

$$U=0,\quad V=0,\quad \frac{\partial J(U,V)}{\partial x}=0,\quad \frac{\partial J(U,V)}{\partial y}=0.$$

19. If

$$(A_0, A_1, A_2, \ldots A_p)(x, y)^p,$$
$$(B_0, B_1, B_2, \ldots B_q)(x, y)^q$$

be two covariants of U_n, prove that the leading coefficient of their Jacobian is

$$pq(A_0B_1 - A_1B_0).$$

20. If

$$(A_0, A_1, A_2, \ldots A_p)(x, y)^p,$$
$$(B_0, B_1, B_2, \ldots B_p)(x, y)^p,$$
$$(C_0, C_1, C_2, \ldots C_p)(x, y)^p$$

be three covariants of U_n, prove that the determinant

$$\begin{vmatrix} A_0 & A_1 & A_2 \\ B_0 & B_1 & B_2 \\ C_0 & C_1 & C_2 \end{vmatrix}$$

is a seminvariant.

CHAPTER XVII.

COVARIANTS AND INVARIANTS OF THE QUADRATIC, CUBIC, AND QUARTIC.

179. **The Quadratic.**—*The quadratic has only one invariant, and no covariant other than the quadratic itself.*

For, if α and β be the roots of the quadratic equation

$$U \equiv ax^2 + 2bx + c = 0,$$

the only functions of their difference which can lead to an invariant or covariant are powers of $\alpha - \beta$ of the type $(\alpha - \beta)^{2p}$; the odd powers of $\alpha - \beta$ not being expressible by the coefficients in a rational form. Whence, expressing

$$U^{2p}\left(\frac{1}{\alpha - x} - \frac{1}{\beta - x}\right)^{2p}$$

by the coefficients, we conclude that the quadratic has only the one distinct invariant $ac - b^2$, and no covariant distinct from U itself.

180. **The Cubic and its Covariants.**—In the present Article the covariants of the cubic will be discussed as examples of the principles already explained, and in the following Article the definite number of covariants and invariants will be determined.

In the case of the cubic a covariant is obtained from a function of the differences of the roots most simply by substituting

$$\beta\gamma + \alpha x,\ \gamma\alpha + \beta x,\ \alpha\beta + \gamma x \text{ for } \alpha, \beta, \gamma \text{ respectively,}$$

and thus avoiding fractions; for, transforming $\alpha - \beta$, we have

$$\frac{1}{\alpha - x} - \frac{1}{\beta - x} = \frac{(\beta\gamma + \alpha x) - (\gamma\alpha + \beta x)}{(\alpha - x)(\beta - x)(\gamma - x)}, \text{ \&c., \&c.,}$$

and when fractions are removed, we arrive at the above transformation (the order being equal to the weight in the case of either function of the differences H or G). This mode of transforming functions of the differences will now be applied to the covariants of the cubic.

(1). *The Quadratic Covariant, or Hessian H_x.*

Transforming both sides of the equation

$$a_0^2(\alpha + \omega\beta + \omega^2\gamma)(\alpha + \omega^2\beta + \omega\gamma) = 9(a_1^2 - a_0a_2),$$

we have

$$a_0^2\{(\alpha + \omega\beta + \omega^2\gamma)x + \beta\gamma + \omega\gamma\alpha + \omega^2\alpha\beta\}$$
$$\times\{(\alpha + \omega^2\beta + \omega\gamma)x + \beta\gamma + \omega^2\gamma\alpha + \omega\alpha\beta\} = 9(U_2^2 - U_3U_1);$$

thus showing that

$$Lx + L_1 \text{ and } Mx + M_1 \qquad \text{(Art. 59)}$$

are the factors of

$$H_x \equiv (a_0a_2 - a_1^2)x^2 + (a_0a_3 - a_1a_2)x + (a_1a_3 - a_2^2),$$

where

$$L_1 \equiv \beta\gamma + \omega\gamma\alpha + \omega^2\alpha\beta, \quad M_1 \equiv \beta\gamma + \omega^2\gamma\alpha + \omega\alpha\beta.$$

From the form of the Hessian in terms of the roots in Art. 167, or from the relations of Art. 43, we conclude that *when a cubic is a perfect cube, each of the coefficients of the Hessian vanishes identically.*

(2). *The Cubic Covariant G_x.*

We have, as in Art. 59,

$$a_0^3\{(\alpha+\omega\beta+\omega^2\gamma)^3+(\alpha+\omega^2\beta+\omega\gamma)^3\} = -27(a_0^2a_3+2a_1^3-3a_0a_1a_2).$$

Transforming both sides of this equation as before, we find

$$a_0^3\{(Lx + L_1)^3 + (Mx + M_1)^3\} = -27(U^2U_0 + 2U_2^3 - 3U_1U_2U)$$
$$= 27G_x,$$

where G_x denotes the covariant formed from the function of differences G operating as in Art. 169 on the source derived from G (the sign being changed in order that G may be the leading coefficient); hence (Ex. 3, Art. 169)

$$G_x = (a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3)x^3 + 3(a_0a_1a_3 + a_1^2a_2 - 2a_0a_1^2)x^2$$
$$-(a_3^2a_0 - 3a_3a_2a_1 + 2a_2^3) - 3(a_3a_2a_0 + a_2^2a_1 - 2a_3a_1^2)x.$$

Resolving $(Lx + L_1)^3 + (Mx + M_1)^3$, we may obtain the factors of G_x; or, more simply, since the factors of G are $\beta + \gamma - 2\alpha$, $\gamma + \alpha - 2\beta$, $\alpha + \beta - 2\gamma$, the factors of G_x are

$$\frac{1}{\beta - x} + \frac{1}{\gamma - x} - \frac{2}{\alpha - x}, \quad \frac{1}{\gamma - x} + \frac{1}{\alpha - x} - \frac{2}{\beta - x}, \quad \frac{1}{\alpha - x} + \frac{1}{\beta - x} - \frac{2}{\gamma - x},$$

when fractions are removed.

We have obviously the following geometrical interpretation of the equation $G_x = 0$: If three points A, B, C determined by the equation $U = 0$ be taken on a right line; and three points A', B', C', such that A' is the harmonic conjugate of A with regard to B and C, B' of B with regard to C and A, and C' of C with regard to A and B; the points A', B', C' are determined by the equation $G_x = 0$. (Compare Ex. 13, p. 88, Vol. I.)

(3). *Expression of the Cubic as the difference of two cubes.*

This can be effected by means of the factors of the Hessian, as follows:—

$$(Lx + L_1)^3 - (Mx + M_1)^3 = 27U\frac{\sqrt{\Delta}}{a_0^3}.$$

For, as in Ex. 6, p. 116, Vol. I., we have

$$L^3 - M^3 = \sqrt{-27}\,(\beta - \gamma)(\gamma - \alpha)(\alpha - \beta).$$

Transforming this equation as before, the first side becomes

$$(Lx + L_1)^3 - (Mx + M_1)^3,$$

and the second side

$$\sqrt{-27}\,(\beta - \gamma)(\gamma - \alpha)(\alpha - \beta)(x - \alpha)(x - \beta)(x - \gamma).$$

Substituting from previous equations, we have

$$(Lx + L_1)^3 - (Mx + M_1)^3 = 27\frac{U}{a_0^4}\sqrt{G^2 + 4H^3} = 27\frac{U\sqrt{\Delta}}{a_0^3}.$$

(4). *Relation between the Cubic and its Covariants.*

The following relation exists:—

$$G_x^2 + 4H_x^3 = \Delta U^2.$$

For, from Ex. 6, p. 116, Vol. I.,

$$a_0^6 (\beta - \gamma)^2 (\gamma - \alpha)^2 (\alpha - \beta)^2 = -27 (G^2 + 4H^3) = -27a_0^2\Delta,$$

and transforming this equation as before,

$$a_0^6 (\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2(x-\alpha)^2(x-\beta)^2(x-\gamma)^2 = -27 (G_x^2+4H_x^3);$$

whence $$\Delta U^2 = G_x^2 + 4H_x^3.$$

This also follows at once by substituting U_n, U_{n-1}, &c., for a_0, a_1, &c., in the identity $G^2 + 4H^3 = a_0^2\Delta$.

(5). *Solution of the Cubic.*

The expression

$$(U\sqrt{\Delta} + G_x)^{\frac{1}{3}} + (U\sqrt{\Delta} - G_x)^{\frac{1}{3}}$$

is a linear factor of U.

For from the relations in (2) and (3) we have

$$2a_0^3(Lx + L_1)^3 = 27(U\sqrt{\Delta} + G_x),$$
$$-2a_0^3(Mx + M_1)^3 = 27(U\sqrt{\Delta} - G_x);$$

and since

$$(Lx + L_1) - (Mx + M_1)$$

is a factor of U, the proposition follows.

This form of solution of the cubic is due to Prof. Cayley.

181. **Number of Covariants and Invariants of the Cubic.**—Before proceeding to the discussion of the quartic, we take up the problem referred to in Art. 162, viz., the determination of the number of independent covariants and invariants, for which purpose we have in the case of the cubic the following proposition:—

The cubic has only two covariants, their leading terms being H *and* G; *and only one invariant, viz. the discriminant* Δ, *where*

$$a^2\Delta = G^2 + 4H^3, \text{ or } \Delta = a^2d^2 + 4ac^3 - 6abcd + 4db^3 - 3b^2c^2.$$

The proof of this can be derived immediately from the proposition of Art. 162. Let $\phi(\alpha, \beta, \gamma)$ be any integral symmetric function of the *differences* of the roots (of order ϖ), expressible by the coefficients in a rational form. It is proved in the proposition referred to that $a^\varpi\phi$ is of the form

$$GF(a, H, \Delta), \text{ or } F(a, H, \Delta),$$

according as ϕ is an odd or even function of the roots. It follows, therefore, in the first place that there cannot be an invariant of an odd degree in the roots, since $GF(a, H, \Delta)$ does not remain the same function when a, b, c, d are changed into d, c, b, a, respectively; and the only invariant of an even degree must be a power of Δ, since if $F(a, H, \Delta)$ contained a or H besides Δ, it could not remain the same function when the coefficients are similarly interchanged.

Again, the cubic has only two distinct covariants; for it has been proved that every seminvariant $a^{\varpi}\phi$ is of one of the forms

$$F(a, H, \Delta), \quad \text{or} \quad GF(a, H, \Delta);$$

and therefore the corresponding covariant, formed from the seminvariant as leading term, must be expressible as

$$F(U, H_x, \Delta), \quad \text{or} \quad G_x F(U, H_x, \Delta);$$

that is, every covariant is expressible in a rational and integral form in terms of H_x and G_x, along with U and Δ; or, in other words, there are only two distinct covariants.

182. **The Quartic. Its Covariants and Invariants.**—We have shown already that the quartic has two invariants, I and J (Art. 167). From the functions H and G of the differences of the roots we can derive two covariants H_x and G_x, whose leading coefficients are H and G; for from the relation

$$a_0^2 \Sigma (\alpha - \beta)^2 = -48(a_0 a_2 - a_1^2)$$

we derive, by the process of Art. 167,

$$a_0^2 \Sigma (\alpha - \beta)^2 (x - \gamma)^2 (x - \delta)^2 = -48(UU_2 - U_3^2) = -48 H_x;$$

and, expanding $UU_2 - U_3^2$, we have

$$H_x \equiv (a_0 a_2 - a_1^2) x^4 + 2(a_0 a_3 - a_1 a_2) x^3 + (a_0 a_4 + 2a_1 a_3 - 3a_2^2) x^2 + 2(a_1 a_4 - a_2 a_3) x + (a_2 a_4 - a_3^2).$$

In a similar manner, since

$$G \equiv a_0^2 a_3 + 2a_1^3 - 3a_0 a_1 a_2,$$

we obtain the covariant

$$-G_x \equiv U^2U_1 + 2U_3^3 - 3UU_3U_2,$$

which reduces to the sixth degree; and if it be written as follows:

$$G_x \equiv A_0x^6 + A_1x^5 + A_2x^4 + A_3x^3 + A_4x^2 + A_5x + A_6,$$

we find, by expanding the above, or more simply, by forming the source A_6, and performing the successive operations of Art. 169, the following values of the coefficients:—

$A_6 = -a_4^2a_1 + 3a_4a_3a_2 - 2a_3^3$, $A_5 = -a_4^2a_0 - 2a_4a_3a_1 - 6a_3^2a_2 + 9a_4a_2^2$,
$A_4 = -5a_4a_3a_0 - 10a_3^2a_1 + 15a_4a_2a_1$, $A_3 = -10a_0a_3^2 + 10a_1^2a_4$,
$A_2 = 5a_0a_1a_4 + 10a_1^2a_3 - 15a_0a_2a_3$, $A_1 = a_0^2a_4 + 2a_0a_1a_3 + 6a_1^2a_2 - 9a_0a_2^2$,
$A_0 = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3$.

Here it will be observed that, when A_3 is determined, A_2, A_1, A_0 may be obtained from A_4, A_5, and A_6 by changing the suffixes into their complementary values, and altering the sign of the whole, in accordance with what was proved in Art. 168.

We proceed in the following Articles to discuss the leading properties of these two covariants of the quartic.

183. **Quadratic Factors of the Sextic Covariant.***— As the quadratic factors of G_x enter prominently into the following discussion, we proceed in the first place to find expressions for those factors in terms of the roots of the quartic, and to deduce their principal properties.

Since the factors of G, expressed in terms of $\alpha, \beta, \gamma, \delta$, are

$$\beta + \gamma - \alpha - \delta, \quad \gamma + \alpha - \beta - \delta, \quad \alpha + \beta - \gamma - \delta,$$

the factors of G_x are obtained from these by substituting $\frac{1}{x-\alpha}, \frac{1}{x-\beta}, \frac{1}{x-\gamma}, \frac{1}{x-\delta}$, for $\alpha, \beta, \gamma, \delta$ respectively, and multiplying each factor by $\frac{U}{a}$ to remove fractions.

* See a Paper by Prof. Ball, *Quarterly Journal of Mathematics*, vol. vii., p. 368, containing a full and valuable discussion of the various solutions of the biquadratic.

Whence, denoting these factors by u, v, w, we have

$$\left.\begin{aligned} au &= U\left(\frac{1}{x-\beta}+\frac{1}{x-\gamma}-\frac{1}{x-\alpha}-\frac{1}{x-\delta}\right),\\ av &= U\left(\frac{1}{x-\gamma}+\frac{1}{x-\alpha}-\frac{1}{x-\beta}-\frac{1}{x-\delta}\right),\\ aw &= U\left(\frac{1}{x-\alpha}+\frac{1}{x-\beta}-\frac{1}{x-\gamma}-\frac{1}{x-\delta}\right),\end{aligned}\right\} \quad (1)$$

which values of u, v, w, arranged in powers of x, are

$$\left.\begin{aligned} u &= (\beta+\gamma-\alpha-\delta)x^2-2(\beta\gamma-\alpha\delta)x+\beta\gamma(\alpha+\delta)-\alpha\delta(\beta+\gamma),\\ v &= (\gamma+\alpha-\beta-\delta)x^2-2(\gamma\alpha-\beta\delta)x+\gamma\alpha(\beta+\delta)-\beta\delta(\gamma+\alpha),\\ w &= (\alpha+\beta-\gamma-\delta)x^2-2(\alpha\beta-\gamma\delta)x+\alpha\beta(\gamma+\delta)-\gamma\delta(\alpha+\beta)\end{aligned}\right\}; \quad (2)$$

and, consequently, $32G_x = a^3uvw$.

From equations (1) we easily find

$$v = (\alpha-\delta)(x-\beta)(x-\gamma)-(\beta-\gamma)(x-\alpha)(x-\delta),$$

$$w = (\alpha-\delta)(x-\beta)(x-\gamma)+(\beta-\gamma)(x-\alpha)(x-\delta);$$

and from these and similar equations we have

$$\frac{v^2-w^2}{\mu-\nu}=\frac{w^2-u^2}{\nu-\lambda}=\frac{u^2-v^2}{\lambda-\mu}=4\frac{U}{a}, \quad (3)$$

where λ, μ, ν have the usual meaning (Ex. 17, Art. 27); and consequently,

$$(\mu-\nu)u^2 = (\lambda-\nu)v^2-(\lambda-\mu)w^2;$$

whence

$$(\mu-\nu)u^2 = (v\sqrt{\lambda-\nu}+w\sqrt{\lambda-\mu})(v\sqrt{\lambda-\nu}-w\sqrt{\lambda-\mu}).$$

Since, as this identical equation shows, the factors on the second side are both perfect squares, we may assume

$$v\sqrt{\lambda-\nu}+w\sqrt{\lambda-\mu} \equiv 2u_1^2,$$

$$v\sqrt{\lambda-\nu}-w\sqrt{\lambda-\mu} \equiv 2u_2^2;$$

we have, therefore,

$$w\sqrt{\lambda-\mu} = u_1^2 - u_2^2,$$
$$v\sqrt{\lambda-\nu} = u_1^2 + u_2^2,$$
$$u\sqrt{\mu-\nu} = 2u_1u_2;$$

from which values we conclude that u, v, w, *the quadratic factors of* G_x, *are mutually harmonic.*

For the geometrical interpretation of the equation $G_x = 0$, see Art. 65.

184. **Expression of the Hessian by the Quadratic Factors of G_x.**—Since

$$-48\frac{H_x}{a^2} = \Sigma(\alpha-\beta)^2(x-\gamma)^2(x-\delta)^2;$$

combining the terms in pairs, and noticing that

$$\Sigma(\beta-\gamma)(\alpha-\delta)U \equiv 0,$$

$$\Sigma(\alpha-\beta)^2(x-\gamma)^2(x-\delta)^2$$
$$= \Sigma\{(\beta-\gamma)(x-\alpha)(x-\delta)+(\alpha-\delta)(x-\beta)(x-\gamma)\}^2,$$

the quantities between brackets being u, v, w, we have

$$-48\frac{H_x}{a^2} = u^2+v^2+w^2,$$

which is the required expression for H_x.

185. **Expression of the Quartic itself by the Quadratic Factors of G_x.**—From equations (3) a symmetrical value may be obtained for U; for, substituting in those equations in place of λ, μ, ν their values in terms of the roots ρ_1, ρ_2, ρ_3 of the equation $4\rho^3 - I\rho + J = 0$, we find

$$a^2(v^2-w^2) = 16(\rho_2-\rho_3)U,\quad a^2(w^2-u^2) = 16(\rho_3-\rho_1)U,$$
$$a^2(u^2-v^2) = 16(\rho_1-\rho_2)U,$$

from which equations, by means of the value of H_x in the preceding Article, we obtain

$$(au)^2 = 16(\rho_1U - H_x),\quad (av)^2 = 16(\rho_2U - H_x), \qquad (4)$$
$$(aw)^2 = 16(\rho_3U - H_x).$$

We now make the substitutions

$$u^2 = \Delta_1 X^2, \quad v^2 = \Delta_2 Y^2, \quad w^2 = \Delta_3 Z^2,$$

where Δ_1, Δ_2, Δ_3 are the discriminants of u, v, w; thus replacing u, v, w by three quadratics X, Y, Z whose discriminants are each equal to unity. By means of this transformation the forms of the quadratics are further fixed, and the identical relation connecting their squares (see (I), Ex. 6, p. 137) is expressed in its simplest form. Calculating their discriminants, we find

$$\Delta_1 = (\beta + \gamma - \alpha - \delta)\{\beta\gamma(\alpha + \delta) - \alpha\delta(\beta + \gamma)\} - (\beta\gamma - \alpha\delta)^2,$$

with similar values of Δ_2 and Δ_3; whence we have

$$\Delta_1 = -(\lambda - \mu)(\lambda - \nu),\ \Delta_2 = -(\mu - \nu)(\mu - \lambda),\ \Delta_3 = -(\nu - \lambda)(\nu - \mu).$$

Making these substitutions, the preceding equations become

$$\begin{aligned} (\rho_1 - \rho_2)(\rho_1 - \rho_3) X^2 &= H_x - \rho_1 U, \\ (\rho_2 - \rho_3)(\rho_2 - \rho_1) Y^2 &= H_x - \rho_2 U, \\ (\rho_3 - \rho_1)(\rho_3 - \rho_2) Z^2 &= H_x - \rho_3 U, \end{aligned} \qquad (5)$$

from which are easily deduced the following values of U and H_x, and the identical equation connecting X, Y, Z:—

$$\begin{aligned} H_x &= \rho_1^2 X^2 + \rho_2^2 Y^2 + \rho_3^2 Z^2, \\ -U &= \rho_1 X^2 + \rho_2 Y^2 + \rho_3 Z^2, \\ 0 &\equiv X^2 + Y^2 + Z^2; \end{aligned} \qquad (6)$$

where, as has been proved, X, Y, Z are three mutually harmonic quadratics whose discriminants are reduced to unity in each case. The value of G_x may be expressed in terms of X, Y, Z as follows. Since $32G_x = a^3 uvw$, and

$$u^2v^2w^2 = (\mu - \nu)^2(\nu - \lambda)^2(\lambda - \mu)^2 X^2Y^2Z^2 = \frac{256}{a^6}(I^3 - 27J^2)X^2Y^2Z^2,$$

we find

$$G_x = \tfrac{1}{2}\sqrt{I^3 - 27J^2} \,.\, XYZ.$$

186. **Resolution of the Quartic.**—From the equations

$$-U = \rho_1 X^2 + \rho_2 Y^2 + \rho_3 Z^2,$$
$$0 = X^2 + Y^2 + Z^2,$$

we find

$$U = (\rho_1 - \rho_2) Y^2 + (\rho_1 - \rho_3) Z^2, \quad U = (\rho_2 - \rho_3) Z^2 + (\rho_2 - \rho_1) X^2,$$
$$U = (\rho_3 - \rho_1) X^2 + (\rho_3 - \rho_2) Y^2,$$

where X^2, Y^2, Z^2 have the values determined by equations (5); and breaking up these values of U into their factors, we have three ways of resolving U depending on the solution of the equation

$$4\rho^3 - I\rho + J = 0.$$

The resolution of the quartic has been presented by Professor Cayley in a symmetrical form which may be easily derived from the expressions already given for U and H_x. For, since in general

$$l(a_1x^2 + 2b_1xy + c_1y^2) + m(a_2x^2 + 2b_2xy + c_2y^2) + n(a_3x^2 + 2b_3xy + c_3y^2)$$

is a perfect square when

$$\Sigma l^2 (a_1c_1 - b_1^2) + \Sigma mn (a_2c_3 + a_3c_2 - 2b_2b_3) = 0,$$

$lX + mY + nZ$ is a perfect square when $l^2 + m^2 + n^2 = 0$,

X, Y, Z being mutually harmonic, and the discriminants each reduced to unity.

The resolution of U is therefore reduced to finding values of l, m, n such that the general quadratic $lX + mY + nZ$, or

$$l\sqrt{\rho_2 - \rho_3}\sqrt{H_x - \rho_1 U} + m\sqrt{\rho_3 - \rho_1}\sqrt{H_x - \rho_2 U}$$
$$+ n\sqrt{\rho_1 - \rho_2}\sqrt{H_x - \rho_3 U},$$

shall be a perfect square, and shall vanish when U vanishes. These values, corresponding to any root $x = \alpha$, may be found by taking any set of values for $\sqrt{\rho_2 - \rho_3}$, $\sqrt{\rho_3 - \rho_1}$, $\sqrt{\rho_1 - \rho_2}$, and a set of values for the square roots of $H_x - \rho_1 U$, $H_x - \rho_2 U$, $H_x - \rho_3 U$, which are each perfect squares, so that the square roots have each the same value for $x = a$, and then taking for X, Y, Z the definite values

$$X = \sqrt{\rho_2 - \rho_3}\sqrt{H_x - \rho_1 U} \,/\, \sqrt{\rho_2 - \rho_3} \,.\, \sqrt{\rho_3 - \rho_1} \,.\, \sqrt{\rho_1 - \rho_2}$$

with similar values for Y, Z, accordingly l, m, n have to satisfy

$$l\sqrt{\rho_2 - \rho_3} + m\sqrt{\rho_3 - \rho_1} + n\sqrt{\rho_1 - \rho_2} = 0,\ l^2 + m^2 + n^2 = 0,$$

which equations are plainly satisfied if

$$l \,/\, \sqrt{\rho_2 - \rho_3} = m \,/\, \sqrt{\rho_3 - \rho_1} = n \,/\, \sqrt{\rho_1 - \rho_2}.$$

Finally, the squares of the four linear factors of U must be

$$(\rho_2 - \rho_3)\sqrt{H_x - \rho_1 U} \pm (\rho_3 - \rho_1)\sqrt{H_x - \rho_2 U} \pm (\rho_1 - \rho_2)\sqrt{H_x - \rho_3 U},$$

of which the product is ΔU^2.

If it be required to solve the quartic $\kappa U - \lambda H_x$, we may similarly select values of l, m, n so that $lX + mY + nZ$ shall be a perfect square and shall vanish when $\kappa U - \lambda H_x$ vanishes. These values may be found by taking a definite set of values for $\sqrt{\rho_2 - \rho_3}$, $\sqrt{\rho_3 - \rho_1}$, $\sqrt{\rho_1 - \rho_2}$, $\sqrt{\kappa}$, writing

$$H_x - \rho_1 U = \frac{\kappa - \rho_1\lambda}{\kappa}\{H_x - \mu_1(\kappa U - \lambda H_x)\}$$

where $\mu_1 = \rho_1 \,/\, (\kappa - \lambda\rho_1)$ with similar values for $H_x - \rho_2 U$, $H_x - \rho_3 U$, selecting values for the square roots of

$$H_x - \mu_1(\kappa U - \lambda H_x), \quad H_x - \mu_2(\kappa U - \lambda H_x), \quad H_x - \mu_3(\kappa U - \lambda H_x),$$

which are each perfect squares, so that they may each have the same value for a definite root a' of $\kappa U - \lambda H_x$, putting

$$X = \sqrt{\rho_2 - \rho_3}\sqrt{\kappa - \lambda\rho_1}\sqrt{H_x - \mu_1(\kappa U - \lambda H_x)} \,/\, \sqrt{\kappa}\sqrt{\rho_2 - \rho_3}\sqrt{\rho_3 - \rho_1}\sqrt{\rho_1 - \rho_2}$$

with similar values for Y, Z; then l, m, n have to satisfy the equations

$$l\sqrt{\rho_2 - \rho_3}\sqrt{\kappa - \lambda\rho_1} + m\sqrt{\rho_3 - \rho_1}\sqrt{\kappa - \lambda\rho_2} + n\sqrt{\rho_1 - \rho_2}\sqrt{\kappa - \lambda\rho_3} = 0$$

$$l^2 + m^2 + n^2 = 0,$$

which are plainly satisfied if

$$l / \sqrt{\rho_2 - \rho_3}\sqrt{\kappa - \lambda\rho_1} = m / \sqrt{\rho_3 - \rho_1}\sqrt{\kappa - \lambda\rho_2} = n / \sqrt{\rho_1 - \rho_2}\sqrt{\kappa - \lambda\rho_3},$$

whence $lX + mY + nZ$ is the square of a linear factor of

$$\kappa U - \lambda H_x = 0.$$

187. **The Invariants and Covariants of** $\kappa U - \lambda H_x$.—Employing the equations (6) of Art. 185, and denoting $X^2 + Y^2 + Z^2$ by V, we may, by adding $-\frac{\lambda I}{6} V$ to $\lambda H_x - \kappa U$, reduce it to the form $R_1X^2 + R_2Y^2 + R_3Z^2$, where $R_1 + R_2 + R_3 = 0$. When this is done, we have the following reduced values of R_1, R_2, R_3:—

$$3R_1 = \kappa(2\rho_1 - \rho_2 - \rho_3) + \lambda(2\rho_2\rho_3 - \rho_3\rho_1 - \rho_1\rho_2),$$
$$3R_2 = \kappa(2\rho_2 - \rho_3 - \rho_1) + \lambda(2\rho_3\rho_1 - \rho_1\rho_2 - \rho_2\rho_3),$$
$$3R_3 = \kappa(2\rho_3 - \rho_1 - \rho_2) + \lambda(2\rho_1\rho_2 - \rho_2\rho_3 - \rho_3\rho_1).$$

On account of the similarity of the forms

$$\rho_1X^2 + \rho_2Y^2 + \rho_3Z^2 \quad \text{and} \quad R_1X^2 + R_2Y^2 + R_3Z^2,$$

which are of the same type, it is clear, and we may verify by direct calculation, that XYZ are also the factors of the sextic covariant of $\kappa U - \lambda H_x$, and that its Hessian is

$$R_1{}^2X^2 + R_2{}^2Y^2 + R_3{}^2Z^2,$$

so that we may calculate the invariants and covariants of $\kappa U - \lambda H_x$ by simply changing ρ_1, ρ_2, ρ_3 into R_1, R_2, R_3 in the expressions for the invariants and covariants of U.

Therefore, since

$$I = \tfrac{2}{3}\{(\rho_2 - \rho_3)^2 + (\rho_3 - \rho_1)^2 + (\rho_1 - \rho_2)^2\}, \quad J = -4\rho_1\rho_2\rho_3,$$

and

$$R_2 - R_3 = (\rho_2 - \rho_3)(\kappa - \lambda\rho_1), \quad R_3 - R_1 = (\rho_3 - \rho_1)(\kappa - \lambda\rho_2),$$
$$R_1 - R_2 = (\rho_1 - \rho_2)(\kappa - \lambda\rho_3),$$

we find the following values for the invariants of $\kappa U - \lambda H_x$:—

$$I_{(\kappa,\lambda)} = I\kappa^2 - 3J\kappa\lambda + \frac{I^2}{12}\lambda^2,$$

$$J_{(\kappa,\lambda)} = J\kappa^3 - \frac{I^2}{6}\kappa^2\lambda + \frac{IJ}{4}\kappa\lambda^2 - \frac{54J^2 - I^3}{216}\lambda^3.$$

If we form the covariants $H_{(\kappa,\lambda)}$, and $G_{(\kappa,\lambda)}$, of Ω, where

$$4\Omega \equiv 4\kappa^3 - I\kappa\lambda^2 + J\lambda^3$$

(the reducing cubic rendered homogeneous in κ, λ), we find, as M. Hermite has remarked,

$$I_{(\kappa,\lambda)} = -12H_{(\kappa,\lambda)},\ J_{(\kappa,\lambda)} = 4G_{(\kappa,\lambda)}.$$

Again, to calculate the Hessian of $\kappa U - \lambda H_x$, we reduce

$$R_1{}^2X^2 + R_2{}^2Y^2 + R_3{}^2Z^2$$

by the substitutions

$$\rho_1{}^3X^2 + \rho_2{}^3Y^2 + \rho_3{}^3Z^2 \equiv -\tfrac{1}{4}IU,$$

$$\rho_1{}^4X^2 + \rho_2{}^4Y^2 + \rho_3{}^4Z^2 \equiv \tfrac{1}{4}(IH_x + JU),$$

which are obtained from the equations

$$\rho_1{}^2 = \rho_2\rho_3 + \tfrac{1}{4}I,\quad \rho_2{}^2 = \rho_3\rho_1 + \tfrac{1}{4}I,\quad \rho_3{}^2 = \rho_1\rho_2 + \tfrac{1}{4}I,$$

by multiplying first by ρ_1X^2, ρ_2Y^2, ρ_3Z^2, respectively, and, secondly, by $\rho_1{}^2X^2$, $\rho_2{}^2Y^2$, $\rho_3{}^2Z_2$, and adding.

In this way we find the following form for the Hessian of $\kappa U - \lambda H_x$:—

$$\frac{1}{4}\left\{H_x\left(4\kappa^2 - \frac{I}{3}\lambda^2\right) - U\left(\frac{2}{3}I\kappa\lambda - J\lambda^2\right)\right\};$$

which may be expressed in the form

$$\frac{1}{3}\left(H_x\frac{\partial\Omega}{\partial\kappa} + U\frac{\partial\Omega}{\partial\lambda}\right),$$

which is a multiple of the Jacobian of $\kappa U - \lambda H_x$ and Ω, the variables being κ and λ.

Again, since $I^3 - 27J^2 = 16(\rho_2 - \rho_3)^2(\rho_3 - \rho_1)^2(\rho_2 - \rho_2)^2$,

and $$G_x = \tfrac{1}{2}\sqrt{I^3 - 27J^2}\,.\,XYZ;$$

transforming ρ_1, ρ_2, ρ_3 into R_1, R_2, R_3, we find

$$I^3_{(\kappa,\lambda)} - 27J^2_{(\kappa,\lambda)} = \Omega^2(I^3 - 27J^2),\quad G_{(\kappa,\lambda)x} = \Omega G_x.$$

We have therefore expressed the invariants and covariants of $\kappa U - \lambda H_x$ in terms of the invariants and covariants of U.

188. **Number of Covariants and Invariants of the Quartic.**—We proceed to prove the following proposition, which determines the number of these functions :—

The quartic has only the two distinct invariants I and J, and two distinct covariants whose leading coefficients are H and G.

This proposition asserts that every invariant is a *rational* and *integral* function of I and J, and every covariant a *rational* and *integral* function of U, H_x, G_x, I, J. The following discussion is founded on principles similar to those already employed in the case of the cubic. It is proved in the proposition of Art. 163, if $\phi(\alpha, \beta, \gamma, \delta)$ be any integral function of the differences of the roots expressible by the coefficients in a rational form, that $a^n\phi(\alpha, \beta, \gamma, \delta)$ may be expressed by the forms

$$GF(a, H, I, J), \quad \text{or} \quad F(a, H, I, J),$$

according as ϕ is odd or even.

Now, if $F(a, H, I, J)$ be an invariant, a and H must disappear, since if they were present this function could not remain the same when the coefficients are written in direct or reverse order. Similarly, no odd function such as $GF(a, H, I, J)$ can give an invariant. It follows, therefore, that every invariant is a function of I and J.

Again, the quartic has only two distinct covariants; for we have proved that every function of the differences $a^n\phi$ is of one of the forms

$$F(a, H, I, J) \quad \text{or} \quad GF(a, H, I, J).$$

Now, considering these forms as the leading coefficients of covariants, it has been proved that every covariant is expressible as

$$F(U, H_x, I, J) \quad \text{or} \quad G_xF(U, H_x, I, J);$$

that is, every covariant is expressible in terms of H_x and G_x, along with U, I, and J; and this is the proposition which was required to be proved.

Examples.

1. If U be any cubic, and G_x its cubic covariant, prove that the Hessian of $\lambda U + \mu G_x$ has the same roots as the Hessian of U, λ and μ being constants.

2. Prove that any covariant of a quantic, whose roots are $a_1, a_2, \ldots a_n$, satisfies the equation

$$\Sigma a^2 \frac{\partial \phi}{\partial a} - \varpi s_1 \phi = x \frac{\partial \phi}{\partial y},$$

where ϖ is the degree of ϕ in the coefficients of the quantic, and $s_1 = \Sigma a$.

3. If a quantic have a square factor, prove that the same square factor enters its Hessian.

4. If a quartic have a square factor, prove that the covariant G_x has that factor as a *quintuple* factor, and give the values of u, v, w in this case. (See Art. 146.)

5. If $\phi(x)$ and $\psi(x)$ be two quantics of the n^{th} degree, the roots of $\phi(x)$ being $a_1, a_2, a_3, \ldots a_n$, show that their Jacobian may be expressed as follows:—

$$J(\phi, \psi) \equiv n\phi^2 \sum_{r=1}^{r=n} \frac{\psi(a_r)}{\phi'(a_r)} \frac{1}{(x - a_r)^2};$$

and in particular prove that the sextic covariant G_x of the quartic $\phi(x)$ may be written under the form

$$\{\phi(x)\}^2 \Sigma \frac{\phi'(a)}{(x-a)^2}.$$

6. Prove for a quartic U that the sum of any two of the quadrics

$$\frac{(x-\alpha)^2}{\phi'(\alpha)}, \quad \frac{(x-\beta)^2}{\phi'(\beta)}, \quad \frac{(x-\gamma)^2}{\phi'(\gamma)}, \quad \frac{(x-\delta)^2}{\phi'(\delta)}$$

is a factor of the sextic covariant of U expressed in terms of the roots.

7. If Δ be the discriminant of

$$\phi(x) \equiv a_0 (x - a_1)(x - a_2)(x - a_3) \ldots (x - a_n),$$

prove that the equation which has for roots the n values of the irrational covariant

$$z_r \equiv \frac{\sqrt{\Delta}}{\phi'(a_r)} (x - a_r)^{n-2}$$

can be expressed in terms of the covariants and invariants of $\phi(x)$ in a *rational* form when $\sqrt{\Delta}$ is adjoined; and show that the values of z when $n = 3$ and $n = 4$, respectively, are multiples of the solutions of the cubic and quartic given by Cayley (Arts. 180, 186).

8. Applying the principles of Art. 188, determine without calculation the form of the sextic covariant of the quartic $\lambda U + \mu H_x$.

9. Calculate the values of H, I, G, J for the Hessian of a quartic.

$$Ans.\quad H' = \frac{3a_0J - HI}{12},\ I' = \frac{I^2}{12},\ G' = -\frac{JG}{4},\ J' = \frac{54J^3 - I^3}{216}.$$

10. Find the two conditions that the Hessian of the quartic wanting its second term should be a perfect square, and show that both contain J as a factor.

$$Ans.\quad JG = 0,\quad a_0J(2HI - 3a_0J) = 0.$$

11. A seminvariant of the equation

$$(a_0, a_1, a_2, \ldots a_n)(x, 1)^n = 0$$

arranged in powers of a_n being

$$\phi \equiv A_p + pA_{p-1}a_n + \frac{p \cdot p-1}{1 \cdot 2} A_{p-2}a_n^2 + \ldots + A_0a_n^p;$$

prove that $DA_j = -na_{n-1}jA_{j-1}$, and hence show that if $\psi(a_0, a_1, a_2, \ldots a_r)$ be a seminvariant, so also is $\psi(A_0, A_1, A_2, \ldots A_r)$.

12. Hence show how the final coefficient of the equation of squared differences can be found for any equation when it is known for the equation of next lower order.

13. If
$$\phi(a_n) \equiv (A_0, A_1, A_2, \ldots A_p)(a_n, 1)^p,$$
$$\psi(a_n) \equiv (B_0, B_1, B_2, \ldots B_q)(a_n, 1)^q$$

be two seminvariants of U_n arranged in powers of a_n, prove that any seminvariant of the system $\phi(x)$ and $\psi(x)$ is a seminvariant of U_{n-1}.

14. If

$$I_1 \equiv (A_0, A_1, A_2, \ldots A_p)(a_n, 1)^p,$$
$$I_2 \equiv (B_0, B_1, B_2, \ldots B_q)(a_n, 1)^q$$

be two invariants of $U_n \equiv (a_0, a_1, a_2, \ldots a_n)(x, y)^n$, prove that the resultant of I_1 and I_2 when a_n is eliminated is the leader of a covariant of U_{n-1} of the degree

$$(n+1)pq - p\pi_2 - q\pi_1$$

in the variables, π_1 and π_2 being the orders of I_1 and I_2.

15. If the discriminant of a biquadratic be written under the form

$$(A_0, A_1, A_2, A_3)(a_4, 1)^3,$$

prove that the discriminant of this cubic is

$$27^2G^2\Delta_3^3,$$

where Δ_3 is the discriminant of $(a_0, a_1, a_2, a_3)(x, 1)^3$: and knowing A_3, find A_2, A_1, and A_0.

16. Form the equation whose roots are

$$\phi(a_1), \quad \phi(a_2), \quad \phi(a_3), \; . \; . \; . \; \phi(a_n),$$

where $a_1, a_2, a_3, \ldots a_n$ are the roots of $f(x) = 0$, the resultant R of $f(x)$ and $\phi(x)$ being given.

Change the last coefficient b_m of $\phi(x)$ into $b_m - \rho$, and substitute this value for b_m in the equation $R = 0$.

17. If $\phi(x) \equiv (a_0, a_1, a_2, \ldots a_n)(x, 1)^n$, whose roots are $a_1, a_2, \ldots a_n$; and if

$$\psi(x) \equiv A_0(x - \beta_1)(x - \beta_2) \; . \; . \; . \; (x - \beta_{n-2}),$$

a covariant of $\phi(x)$ of the degree $n - 2$, prove that any symmetric function of the quantities

$$\frac{\psi(a_1)}{\phi'(a_1)}, \quad \frac{\psi(a_2)}{\phi'(a_2)}, \; . \; . \; . \; \frac{\psi(a_n)}{\phi'(a_n)}$$

can be expressed by invariants. (HERMITE.)

Denoting by R the resultant of $\phi(x)$ and $\lambda\phi'(x) + \psi(x)$ expressed in terms of the roots of $\phi(x)$, we can prove this proposition simply by showing that the values of λ given by the equation $R = 0$ are not altered (except in sign) when for the roots a and the roots β their reciprocals are written; the inversion of the roots a involving the substitution of a_{n-r} for a_r and also the inversion of the roots β of $\psi(x)$.

18. Prove that U_4 and H_x expressed in terms of u_1 and u_2 of Art. 183 are both of the form

$$(A, B, A)(u_1{}^2, u_2{}^2)^2.$$

19. Prove that the quartic

$$f(x, y) \equiv (a, b, c, d, e)(x, y)^4$$

may be reduced by a linear transformation $x = \lambda X + \mu Y$, $y = \lambda' X + \mu' Y$ to the form

$$f(\lambda, \lambda')X^4 + f(\mu, \mu')Y^4 + 6\rho M^2 X^2 Y^2,$$

where

$$4\rho^3 - I\rho + J = 0, \quad M \equiv \lambda\mu' - \lambda'\mu. \qquad \text{(SYLVESTER.)}$$

20. Retaining the notation of the last example, prove that $\dfrac{\lambda}{\lambda'}$ and $\dfrac{\mu}{\mu'}$ are the roots of one of the factors u, v, w of the sextic covariant of the quartic (Art. 183).

21. Prove that

$$\frac{d^3 G_x}{dx^3} = 60(U_1{}^2 U_4 - U_0 U_3{}^2),$$

the reducing cubic of Art. 65 (cf. Ex. 5, p. 132, Vol. I.).

22. Prove that

$$\rho_1{}^p X^2 + \rho_2{}^p Y^2 + \rho_3{}^p Z^2 = \Pi_{p-2} H_x - \Pi_{p-1} U,$$

where Π_{p-1}, Π_{p-2} are sums of homogeneous products.

23. Prove

$$(27J^2 - I^3)(Y^2Z^2 + Z^2X^2 + X^2Y^2) \equiv I^2U^2 - 36JUH_x + 12IH_x^2,$$

the second side of this identity being the Hessian of G_x.

24. If
$$U \equiv \xi^4 + \eta^4 + 6m\xi^2\eta^2,$$
where
$$\xi = \lambda x + \mu y, \quad \eta = \lambda' x + \mu' y, \quad M = \lambda\mu' - \lambda'\mu;$$
prove that
$$I \equiv M^4(1 + 3m^2), \quad J \equiv M^6(m - m^3),$$
$$M^2 = \frac{J}{I}\,\frac{1 + 3m^2}{m - m^3}, \quad 4(M^2m)^3 - I(M^2m) + J = 0,$$
$$H_x \equiv M^2\{m(\xi^4 + \eta^4) + (1 - 3m^2)\xi^2\eta^2\},$$
$$G_x \equiv M^3(1 - 9m^2)\xi\eta(\xi^4 - \eta^4).$$

25. Show (1) that there are two real and distinct ways of making the transformation of Example 19 when the roots of the quartic are all real or all imaginary, and (2) only one real way when two roots are real and two imaginary.

Calculate the values of Δ_1, Δ_2, Δ_3 of Art. 185, and observe that in the first case the reducing cubic has three real roots, and in the second one real and two imaginary roots.

CHAPTER XVIII.

COVARIANTS ND INVARIANTS OF COMBINED FORMS.

189. **Combined Forms.**—In the present chapter we propose to illustrate the theory of the covariants and invariants of systems of two or more quantics (Art. 166) by the simplest cases, viz.: (1) two quadratics, (2) quadratic and cubic, and (3) two cubics. We give in each case an enumeration of the forms which have been shown to be fundamental by the investigations of Clebsch, Gordan, and Sylvester, showing how these forms may be obtained, but without attempting the reduction of all other forms dependent on them. In estimating the number of covariants and invariants of a combined system, the independent forms which belong to each quantic by itself are counted among the total number belonging to the system. It will be found convenient to use the term *special* to designate those forms which belong to the two quantics *regarded as a system* (and which therefore contain the coefficients of both), as distinguished from those which belong to the quantics taken separately.

Invariants and covariants are both included under the name *concomitant*, which is applied to any function whose relations to the quantics are independent of linear transformation.

190. **Two Quadratics.**—Let the two quadratics be

$$U \equiv a_1x^2 + 2b_1xy + c_1y^2, \quad V \equiv a_2x^2 + 2b_2xy + c_2y^2.$$

This system has one special invariant, and one special covariant. The invariant may be obtained by forming the discriminant of $\lambda U + \mu V$, which is found to be

$$\lambda^2(a_1c_1 - b_1^2) + \lambda\mu(a_1c_2 + a_2c_1 - 2b_1b_2) + \mu^2(a_2c_2 - b_2^2),$$

all the coefficients of $\lambda : \mu$ being invariants (Art. 175); whence we have the special invariant

$$a_1c_2 + a_2c_1 - 2b_1b_2 \equiv 2I_{12}. \qquad \text{(Ex. 3, Art. 171.)}$$

The vanishing of this function of the coefficients is the condition that the pencil of lines $UV = 0$ should be harmonic, the rays represented by one equation being conjugate to those represented by the other.

The special covariant is the Jacobian of the system, viz.,

$$\begin{vmatrix} a_1x + b_1y & b_1x + c_1y \\ a_2x + b_2y & b_2x + c_2y \end{vmatrix} \equiv J(U, V),$$

which may be written in the form

$$\begin{vmatrix} y^2 & -xy & x^2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix},$$

obtained by eliminating dialytically the variables from the quantics U, V, $(xy' - x'y)^2$, the form $xy' - x'y$ being a universal concomitant of all binary quantics (Art. 175). This form for $J(U, V)$ can also be arrived at by eliminating λ and μ from the equations obtained by comparing the coefficients in the identity $\lambda U + \mu V \equiv (xy' - x'y)^2$.

The square of J is connected with U and V by the following important relation:—

$$-J^2(U, V) = I_{22}U^2 - 2I_{12}UV + I_{11}V^2, \qquad (1)$$

which may be derived immediately from the equation

$$\begin{vmatrix} y^2 & -xy & x^2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \begin{vmatrix} x^2 & 2xy & y^2 \\ c_1 & -2b_1 & a_1 \\ c_2 & -2b_2 & a_2 \end{vmatrix} = \begin{vmatrix} 0 & U & V \\ U & 2I_{11} & 2I_{12} \\ V & 2I_{12} & 2I_{22} \end{vmatrix}.$$

Again, it is easy to see that $J(UV)$ gives the double lines of the system $\lambda U + \mu V$, for when $\lambda U + \mu V$ is a perfect square

$$\lambda^2 I_{11} + 2\lambda\mu I_{12} + \mu^2 I_{22} = 0,$$

and eliminating $\lambda : \mu$ by means of the equation $\lambda U + \mu V = 0$, the double lines are determined by the equation

$$I_{22}U^2 - 2I_{12}UV + I_{11}V^2 = 0,$$

or

$$J^2(U, V) = 0.$$

Every concomitant of a system of two quadratics may be expressed in terms of the six forms U, V, $J(U, V)$, I_{11}, I_{12}, I_{22} all of which are constituents of the formula (1) written above. The resultant of U, V, for example, is

$$4(I_{11}I_{22} - I_{12}^2), \qquad \text{(Art. 150.)}$$

which is also the discriminant of $J(U, V)$, and the dialytic eliminant of U, V, $J(U, V)$.

191. **Quadratic and Cubic.**—Let the two quantics be

$$U \equiv (a, b, c, d)(x, y)^3, \quad V \equiv (a', b', c')(x, y)^2,$$

the covariants of U being denoted as usual by H_x and G_x. The system has one special cubic covariant, the Jacobian of U and V, or $J(U, V)$; and one special quadratic covariant, viz., $J(H_x, V)$.

In writing down the remaining covariants it will be found convenient to adopt the following notation. We use U with suffix D to denote the result of substituting in U the differential symbols D_y, $-D_x$ for x, y, respectively, where $D_x \equiv \frac{\partial}{\partial x}$, $D_y \equiv \frac{\partial}{\partial y}$; hence

$$U_D \equiv (a, b, c, d)(D_y, -D_x)^3, \quad V_D \equiv (a', b', c')(D_y, -D_x)^2,$$

with a corresponding notation in other cases.

There are four linear covariants, which may now be written as follows :—

$$V_D(U), \quad V_D(G_x), \quad U_D(V^2), \quad G_D(V^2).$$

The first of these written at length is

$$(ac' - 2bb' + ca')x + (bc' - 2cb' + da')y.$$

There are three special invariants. The first is the intermediate invariant of the system of two quadratics H_x and V, viz.,

$$(ac - b^2)c' - (ad - bc)b' + (bd - c^2)a' \equiv I_{21},$$

where the notation $\mathbf{I}_{pq}$ is used to signify that the invariant is of the p^{th} degree in the coefficients of U, and the q^{th} in the coefficients of V. The second invariant is the resultant R of U and V. It is of the second degree in the coefficients of U, and third in the coefficients of V, and may be expressed in many ways by the methods of elimination of Chap. XIV. The general form of any invariant I_{23} of this type is

$$I_{23} \equiv lR + m\,(a'c' - b'^2)I_{21},$$

l and m being any numbers.

The third invariant (which is skew) is of the type I_{43}, and may be obtained by operating with V_D three times in succession on the product of U and G_x; it can be written in the form

$$V_D{}^3(UG_x).$$

There are, therefore, nine special forms belonging to this system; and if to these be added U and V, and the independent covariants and invariants of each, we obtain the complete list of fifteen forms, viz., three cubic, three quadratic, and four linear covariants, and five invariants.

192. **Two Cubics.**—Let the cubics be

$$U \equiv (a, b, c, d)\,(x, y)^3, \quad V \equiv (a', b', c', d')\,(x, y)^3,$$

the covariants of U being represented as before by H_x and G_x and those of V by H'_x and G'_x.

Of this system there is one quartic covariant, the Jacobian of U and V, viz.,

$$J\,(U, V) \equiv (ab')\,x^4 + 2\,(ac')\,x^3y + \{(ad') + 3\,(bc')\}\,x^2y^2 + 2\,(bd')\,xy^3 + (cd')\,y^4\,;$$

and two special cubic covariants, viz.,

$$J\,(U, H'_x), \text{ and } J\,(V, H_x).$$

There are four special quadratic covariants. If we form the Hessian of $\lambda U + \mu V$, i.e. substitute $\lambda a + \mu a'$, $\lambda b + \mu b'$, &c., for a, b, &c., in H_x, we find

$$\lambda^2 H_x + \lambda\mu M_x + \mu^2 H'_x.$$

The intermediate Hessian M_x here obtained is the first special quadratic covariant; and the remaining three are obtained by taking the Jacobians in pairs of H_x, M_x, and H'_x.

There are six linear covariants which may be written as follows:—

$$H_D(V), \quad H_D(G'_x), \quad H'_D(U), \quad H'_D(G_x), \quad U_D(H'^2_x), \quad V_D(H^2_x).$$

It is easily seen that $H_D(U)$ and $H_D(G_x)$ vanish identically, for U and G_x may be brought by linear transformation to the forms $ax^3 + dy^3$, and $ad(ax^3 - dy^3)$, respectively, and H_x to the form $adxy$ (cf. Art. 180).

There are in all seven invariants, five of which may be obtained by forming the discriminant of $\lambda U + \mu V$, the coefficients of the different powers of $\lambda : \mu$ being invariants. If the discriminant is

$$\lambda^4\Delta + 4\lambda^3\mu\Theta + 6\lambda^2\mu^2\Phi + 4\lambda\mu^3\Theta' + \mu^4\Delta',$$

we obtain in this way three special invariants Θ, Φ, Θ', the extreme coefficients being the discriminants of U and V. The two remaining invariants are of odd orders in the coefficients of each cubic. They are denoted by P and Q, and may be defined as follows:—

$$P \equiv \tfrac{1}{6}U_D(V) = (ad') - 3(bc'), \qquad (1)$$

$$27Q \equiv P^3 - R, \qquad (2)$$

where R is the resultant of U and V as obtained by Bezout's method (Art. 155), viz.,

$$R \equiv (ad')^3 - 18(ab')(cd')(ad') + 9(bd')(ca')(ad') + 27(ca')^2(cd') + 27(ab')(bd')^2 - 81(ab')(bc')(cd').$$

Substituting this value of R in (2), we find

$$-Q \equiv (bc')^3 + (ca')^2(cd') + (ab')(bd')^2 - (bc')^2(ad') - 3(ab')(bc')(cd') - (ad')(ab')(cd').$$

Any invariant comprised in the formula $lP^3 + mR$, where l and m are numbers, being of the type I_{33}, might have been selected instead of Q as the fundamental invariant of this type; reasons will appear subsequently for the selection which has been made (see Ex. 4, p. 164).

If to the special forms enumerated be added those which belong to each cubic, we have in all twenty-six fundamental forms, viz., one quartic, six cubic, six quadratic, and six linear, covariants; and seven invariants.

Several of the covariants and invariants enumerated in the preceding Articles will be found expressed in terms of the roots of the two equations of the combined system among the examples which follow on the next page.

193. **Combinants.**—Combined forms of the same degree give rise to a series of invariants and covariants whose coefficients are expressible by determinants of the form $(a_r b_s)$, such as occur in the resultant obtained by Bezout's method (Art. 155). These concomitants are unaltered, save by a factor of the form $(\lambda\mu' - \lambda'\mu)^r$, when the quantics U, V are changed into $\lambda U + \mu V$, $\lambda' U + \mu' V$. Such invariants have been called *combinants*, and the corresponding covariants may be termed in like manner *combining covariants*. Of the former we have examples in P and Q of Art. 192; and the Jacobians of such forms are examples of the latter class of concomitants.

It may be noticed that the I and J invariants of the biquadratic in $\lambda : \mu$ of the preceding Article, viz., the discriminant of $\lambda U + \mu V$, are combinants of the system of two cubics; for, in fact, a linear transformation of λ and μ is equivalent to a transformation of U and V of the kind considered in the present Article, and therefore any function of the invariants Δ, Θ, Φ, &c., unaltered by such transformation, must be a combinant. It can be verified that these invariants may be expressed in terms of P and Q as follows (see Salmon's *Higher Algebra*, Art. 218):—

$$I = 3P(P^3 - 24Q), \quad J = -P^6 + 36P^3Q - 216Q^2.$$

Examples.

1. If α, β, γ, and α', β' are the roots of the equations

$$U \equiv ax^3 + 3bx^2 + 3cx + d = 0, \quad V \equiv a'x^2 + 2b'x + c' = 0,$$

express in terms of the coefficients the function

$$(\beta - \gamma)^2(\alpha - \alpha')(\alpha - \beta') + (\gamma - \alpha)^2(\beta - \alpha')(\beta - \beta') + (\alpha - \beta)^2(\gamma - \alpha')(\gamma - \beta').$$

Denoting this function by ϕ, we easily find

$$-a^2a'\phi = 9\{a'(bd - c^2) - b'(ad - bc) + c'(ac - b^2)\}.$$

The given function of the roots is an invariant of the system, for it involves all the roots of the cubic in the second degree, and all the roots of the quadratic in the first degree. If, in fact, we make the substitutions of Art. 166, and multiply by U^2V to make the function integral, the result will not contain x, and is therefore an invariant (Art. 191).

The geometrical interpretation of the equation $\phi = 0$ is that the quadratic V should form with the Hessian of U a harmonic system.

2. Using the same notation as in the preceding question, find the condition that one pair of roots of $U = 0$ should form a harmonic range with the roots of $V = 0$.

Ans. $R + 9(a'c' - b'^2)I_{21} = 0$.

3. If α, β, γ, and α', β', γ' be the roots of the cubics

$$U \equiv ax^3 + 3bx^2 + 3cx + d = 0, \quad V \equiv a'x^3 + 3b'x^2 + 3c'x + d' = 0,$$

express the following function (when multiplied by aa') in terms of the coefficients, and prove that it is an invariant of the system :—

$$(\alpha - \alpha')(\beta - \beta')(\gamma - \gamma') + (\alpha - \beta')(\beta - \gamma')(\gamma - \alpha') + (\alpha - \gamma')(\beta - \alpha')(\gamma - \beta');$$

or, differently arranged,

$$(\alpha - \alpha')(\beta - \gamma')(\gamma - \beta') + (\alpha - \beta')(\beta - \alpha')(\gamma - \gamma') + (\alpha - \gamma')(\beta - \beta')(\gamma - \alpha');$$

Ans. $3P$, where $P \equiv (ad' - a'd) - 3(bc' - b'c)$. (Art. 192.)

4. Retaining the notation of the preceding example, prove that if κ can be determined so as to make $U + \kappa V$ a perfect cube, the following relation exists among the roots of the two cubics :—

$$(\beta - \gamma)\sqrt[3]{\phi(\alpha)} + (\gamma - \alpha)\sqrt[3]{\phi(\beta)} + (\alpha - \beta)\sqrt[3]{\phi(\gamma)} = 0,$$

where $\phi(x) \equiv V$ and α, β, γ are the roots of $U = 0$; and prove that in this case the invariant Q (Art. 192) vanishes.

The relation among the roots is obtained immediately by substituting α, β, γ for x in the identity $U + \kappa V \equiv (lx + m)^3$, and eliminating κ, l, m from the resulting equations.

Rationalizing, we have

$$\left\{\frac{(\beta-\gamma)^3\phi(\alpha)+(\gamma-\alpha)^3\phi(\beta)+(\alpha-\beta)^3\phi(\gamma)}{(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)}\right\}^3 - 27\,\phi(\alpha)\,\phi(\beta)\,\phi(\gamma) = 0.$$

Substituting for $\phi(\alpha)$, $\phi(\beta)$, $\phi(\gamma)$; introducing the relations obtained by comparing the different powers of λ in the following identity:—

$$\Sigma(\alpha+\lambda)^3(\beta-\gamma)^3 \equiv 3(\alpha+\lambda)(\beta+\lambda)(\gamma+\lambda)(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta);$$

and expressing the result in terms of the coefficients, we find

$$\{3P\}^3 - 27R = 0, \quad \text{or} \quad Q = 0 \text{ (Art. 192)}.$$

We now give several different forms under which the invariant Q presents itself. Since $U + \kappa V$ is a perfect cube, we have (Art. 43)—

$$\frac{a+\kappa a'}{b+\kappa b'} = \frac{b+\kappa b'}{c+\kappa c'} = \frac{c+\kappa c'}{d+\kappa d'}. \qquad (1)$$

Equating these fractions separately to $-\kappa'$, we find

$$\left.\begin{aligned} a+\kappa a'+\kappa' b+\kappa\kappa' b' &= 0, \\ b+\kappa b'+\kappa' c+\kappa\kappa' c' &= 0, \\ c+\kappa c'+\kappa' d+\kappa\kappa' d' &= 0; \end{aligned}\right\} \qquad (2)$$

and solving for κ, κ', $\kappa\kappa'$, we may eliminate them, and find the condition in the form

$$Q \equiv \begin{vmatrix} a & b & c \\ a' & b' & c' \\ b' & c' & d' \end{vmatrix}\begin{vmatrix} b' & c' & d' \\ a & b & c \\ b & c & d \end{vmatrix} - \begin{vmatrix} a' & b' & c' \\ a & b & c \\ b & c & d \end{vmatrix}\begin{vmatrix} b & c & d \\ a' & b' & c' \\ b' & c' & d' \end{vmatrix} = 0.$$

Again, eliminating κ and κ^2 from the equations (1) without introducing κ', we obtain another form for Q, viz.,

$$Q \equiv \begin{vmatrix} ac-b^2 & ac'+a'c-2bb' & a'c'-b'^2 \\ ad-bc & ad'+a'd-bc'-b'c & a'd'-b'c' \\ bd-c^2 & bd'+b'd-2cc' & b'd'-c'^2 \end{vmatrix}.$$

This form of Q can be readily obtained also by expressing the condition that the Hessian of $\lambda U + \mu V$ (Art. 192) should vanish identically—a condition which is fulfilled when $\lambda U + \mu V$ is a perfect cube.

Finally, writing the equations (2) in the form

$$\frac{a+\kappa' b}{a'+\kappa' b'} = \frac{b+\kappa' c}{b'+\kappa' c'} = \frac{c+\kappa' d}{c'+\kappa' d'},$$

and eliminating κ' and κ'^2, we have a third form for Q, viz.,

$$Q \equiv \begin{vmatrix} (ab') & (ac') & (bc') \\ (ac') & (ad')+(bc') & (bd') \\ (bc') & (bd') & (cd') \end{vmatrix}.$$

The constituents in this form are the same minor determinants which occur in Bezout's form of the resultant; and it may be easily verified that this value of Q agrees with the expanded form written in Art. 192.

5. Find the condition that the roots of two cubics should determine a system in involution.

The condition in terms of the roots is expressed by equating to zero the product of six determinants of the type

$$\begin{vmatrix} 1 & \alpha + \alpha' & \alpha\alpha' \\ 1 & \beta + \beta' & \beta\beta' \\ 1 & \gamma + \gamma' & \gamma\gamma' \end{vmatrix}.$$

6. Express the condition of the preceding example in terms of the coefficients of the cubics.

The roots of one cubic being conjugates to the roots of the other, the two are reducible to the following forms:—

$$U \equiv ax^3 + 3bx^2 + 3cx + d,$$
$$V \equiv dx^3 + 3\kappa cx^2 + 3\kappa^2 bx + \kappa^3 a;$$

and writing the discriminant of $\rho U + V$ in general in the form (Art. 192)—

$$\rho^4\Delta + 4\rho^3\Theta + 6\rho^2\Phi + 4\rho\Theta' + \Delta',$$

we find in this case

$$\Theta' = \kappa^3\Theta, \quad \Delta' = \kappa^6\Delta;$$

whence the required condition

$$\Delta\Theta'^2 - \Delta'\Theta^2 = 0.$$

7. Express in terms of the coefficients of the cubics of Ex. 3 the following covariant of the system:—

$$aa'\Sigma\{3(\beta - \beta')(\gamma - \gamma') + 3(\beta - \gamma')(\gamma - \beta') + (\beta - \gamma)(\beta' - \gamma')\}(x - \alpha)(x - \alpha').$$

Ans. $18\{(ac' + a'c - 2bb')x^2 + (ad' + a'd - bc' - b'c)x + (bd' + b'd - 2cc')\}$.

8. To reduce the two cubics

$$U \equiv (a, b, c, d)(x, y)^3, \quad V \equiv (a', b', c', d')(x, y)^3$$

to the forms

$$U = \frac{1}{4}\frac{\partial F}{\partial X}, \quad V = \frac{1}{4}\frac{\partial F}{\partial Y},$$

by means of a linear transformation whose coefficients are to be determined in terms of the coefficients of the given cubics.

If $\qquad F = (A, B, C, D, E)(X, Y)^4;$

and $\qquad U \equiv (a, b, c, d)(x, y)^3 = (A, B, C, D)(X, Y)^3,$

$\qquad V \equiv (a', b', c', d')(x, y)^3 = (B, C, D, E)(X, Y)^3,$

by substituting the differential symbols $D_y, -D_x$ for x and y, and $\frac{1}{M}D_Y$, $-\frac{1}{M}D_X$ for X and Y in the Hessian of both forms of U, we find

$$\begin{vmatrix} D_x^2 & D_xD_y & D_y^2 \\ a & b & c \\ b & c & d \end{vmatrix} = \frac{1}{M^4}\begin{vmatrix} D_X^2 & D_XD_Y & D_Y^2 \\ A & B & C \\ B & C & D \end{vmatrix};$$

whence, operating on both forms of V, we have

$$\psi(x, y) \equiv \begin{vmatrix} a' & b' & c' \\ a & b & c \\ b & c & d \end{vmatrix} x + \begin{vmatrix} b' & c' & d' \\ a & b & c \\ b & c & d \end{vmatrix} y = \frac{JY}{M^4}.$$

Similarly

$$\phi(x, y) \equiv \begin{vmatrix} a & b & c \\ a' & b' & c' \\ b' & c' & d' \end{vmatrix} x + \begin{vmatrix} b & c & d \\ a' & b' & c' \\ b' & c' & d' \end{vmatrix} y = \frac{JX}{M_4},$$

where ϕ and ψ are covariants of U and V, and J is the ternary invariant of F.

Again, since

$$\phi_D = \phi(D_y, -D_x) = \frac{J}{M^5}D_Y, \text{ and } -\psi_D = -\psi(D_y, -D_x) = \frac{J}{M^5}D_X,$$

performing the operation

$$\phi_D\psi(x, y), \quad \text{or} \quad \psi_D\phi(x, y),$$

on equivalent forms, we have

$$Q \equiv \begin{vmatrix} a & b & c \\ a' & b' & c' \\ b' & c' & d' \end{vmatrix}\begin{vmatrix} b' & c' & d' \\ a & b & c \\ b & c & d \end{vmatrix} - \begin{vmatrix} a' & b' & c' \\ a & b & c \\ b & c & d \end{vmatrix}\begin{vmatrix} b & c & d \\ a' & b' & c' \\ b' & c' & d' \end{vmatrix} = \frac{J^2}{M^9}.$$

We are now in a position to prove that U, V may be reduced to the required forms.

From former equations we have

$$Qx = \begin{vmatrix} b' & c' & d' \\ a & b & c \\ b & c & d \end{vmatrix}\phi - \begin{vmatrix} b & c & d \\ a' & b' & c' \\ b' & c' & d' \end{vmatrix}\psi \equiv m\phi - m'\psi.$$

$$Qy = -\begin{vmatrix} a' & b' & c' \\ a & b & c \\ b & c & d \end{vmatrix}\phi + \begin{vmatrix} a & b & c \\ a' & b' & c' \\ b' & c' & d' \end{vmatrix}\psi \equiv -l\phi + l'\psi.$$

$$\psi = lx + my, \quad \phi = l'x + m'y.$$

If, using this transformation,

$$Q^3U = (A', B', C', D')(\phi, \psi)^3, \quad Q^3V = (A'', B'', C'', D'')(\phi, \psi)^3,$$

we have

$$A' = am^3 - 3bm^2l + 3cml^2 - dl^3 = \tfrac{1}{6}\psi_D{}^3U,$$
$$B' = -am^2m' + bm^2l' + 2bmm'l - 2cmll' - cl^2m' + dl^2l'$$
$$= m^2(bl' - am') + 2ml(bm' - cl') + l^2(dl' - cm').$$

Now if the Hessians H, H' of U, V are equal to

$$\gamma x^2 - \beta xy + \alpha y^2, \quad \gamma' x^2 - \beta' xy + \alpha' y^2$$

respectively, we have $l' = a\alpha' + b\beta' + c\gamma'$, $m' = b\alpha' + c\beta' + d\gamma'$, with similar values for l, m, and hence

$$bl' - am' = (\beta\gamma') = a'm - b'l, \quad cl' - bm' = (\gamma\alpha') = b'm - c'l,$$
$$dl' - cm' = (\alpha\beta') = c'm - d'l.$$

Hence $B' = m^2(\beta\gamma') - 2ml(\gamma\alpha') + l^2(\alpha\beta') = \tfrac{1}{2}\psi_D{}^2J(H, H')$,

$$B' \text{ also} = -\tfrac{1}{6}\psi_D{}^2\phi_DU$$

$$C' = mm'(am' - bl') + ml'(cl' - bm') + m'l(cl' - bm') + ll'(cm' - dl')$$
$$= -mm'(\beta\gamma') + (ml' + m'l)(\gamma\alpha') - ll'(\alpha\beta') = -\tfrac{1}{2}\psi_D\phi_DJ(HH'),$$
$$C' \text{ also} = \tfrac{1}{6}\psi_D\phi_D{}^2U$$

$$D' = m'^2(\beta\gamma') - 2m'l'(\gamma\alpha') + l'^2(\alpha\beta) = \tfrac{1}{2}\phi_D{}^2J(H, H'), \quad D' \text{ also} = -\tfrac{1}{6}\phi_D{}^2U.$$

Similarly,

$$A'' = m^2(\beta\gamma') - 2ml(\gamma\alpha') + l^2(\alpha\beta') = B'$$
$$B'' = -mm'(\beta\gamma') + (ml' + m'l)(\gamma\alpha') - ll'(\alpha\beta') = C'$$
$$C'' = m'^2(\beta\gamma') - 2m'l'(\gamma\alpha') + l'(\alpha\beta) = D'$$
$$D'' = -\tfrac{1}{6}\phi_D{}^3V.$$

Hence, putting

$$A' = Q^3A, \quad B' = A'' = Q^3B, \quad C' = B'' = Q^3C, \quad D' = C'' = Q^3D, \quad D'' = Q^3E,$$
$$F = (A, B, C, D, E)(\phi, \psi)^4,$$

we have
$$U = \frac{1}{4}\frac{\partial F}{\partial \phi}, \quad V = \frac{1}{4}\frac{\partial F}{\partial \psi},$$

and we note that A, B, C, D, E are invariants as they should be.

9. Determine the invariants of F in the preceding example, and hence infer the form of the resultants of two cubics.

We have, from the equations of Ex. 8,

$$Q = J^2 / M^9; \; M = (ml') / Q^2 = Q,$$

and, substituting differential symbols for x, y and ϕ, ψ in both forms of V, and operating on U, we find

$$P \equiv ad' - a'd - 3(bc' - b'c) = I/M^3 = I/Q^3.$$

Hence
$$J^2 = Q^{10}, \quad I = Q^3P,$$
$$I^3 - 27J^2 = Q^9(P^3 - 27Q),$$

from which it follows that when $P^3 = 27Q$, we have $I^3 = 27J^2$; but the last relation holds when F has a square factor, which necessitates U and V having a common factor; whence, if $P^3 - 27Q$, U and V have a common factor, and therefore $P^3 - 27Q$, being of proper degree and weight, is the resultant of the cubics U and V (cf. Art. 192).

10. If $\alpha, \beta, \gamma, \delta : \alpha', \beta', \gamma', \delta'$ be the roots of the biquadratics

$$(a, b, c, d, e)(x, 1)^4 = 0, \quad (a', b', c', d', e')(x, 1)^4 = 0,$$

prove

$$aa'\Sigma(\alpha - \alpha')(\beta - \beta')(\gamma - \gamma')(\delta - \delta') = 24\{ae' + a'e - 4(bd' + b'd) + 6cc'\},$$

and show that this function is an invariant of the system.

11. Prove that the following function of the roots of a biquadratic and quadratic gives an invariant of the system, and determine its geometrical interpretation :—

$$\begin{vmatrix} 1 & \beta+\gamma & \beta\gamma \\ 1 & \alpha+\delta & \alpha\delta \\ 1 & \alpha'+\beta' & \alpha'\beta' \end{vmatrix} \times \begin{vmatrix} 1 & \gamma+\alpha & \gamma\alpha \\ 1 & \beta+\delta & \beta\delta \\ 1 & \alpha'+\beta' & \alpha'\beta' \end{vmatrix} \times \begin{vmatrix} 1 & \alpha+\beta & \alpha\beta \\ 1 & \gamma+\delta & \gamma\delta \\ 1 & \alpha'+\beta' & \alpha'\beta' \end{vmatrix} \equiv \phi.$$

The geometrical interpretation of the equation $\phi = 0$ is, that the two conjugate foci of some one of the three involutions determined by the biquadratic form along with the quadratic an harmonic system.

12. Prove that the following functions of the roots of a biquadratic and quadratic give invariants of the system, and determine their values in terms of the coefficients :—

$$a_0 b_0^2 \Sigma(\alpha' - \alpha)(\alpha' - \beta)(\beta' - \gamma)(\beta' - \delta),$$
$$a_0^2 b_0^2 \Sigma(\alpha - \beta)^2(\gamma - \alpha')(\delta - \beta')(\gamma - \beta')(\delta - \alpha').$$

13. If $f(x)$ and $\phi(x)$ be two quartics with unequal roots, the roots of $f(x)$ being $\alpha, \beta, \gamma, \delta$, prove that the condition that a quartic of the system $\lambda f(x) + \mu\phi(x)$ can have two square factors may be expressed as follows :—

$$\begin{vmatrix} 1 & \alpha & \alpha^2 & \sqrt{\phi(\alpha)} \\ 1 & \beta & \beta^2 & \sqrt{\phi(\beta)} \\ 1 & \gamma & \gamma^2 & \sqrt{\phi(\gamma)} \\ 1 & \delta & \delta^2 & \sqrt{\phi(\delta)} \end{vmatrix} = 0.$$

14. Determine the condition in terms of the coefficients that the quartic of the form $\lambda f(x) + \mu\phi(x)$ may have two square factors.

In this case the Hessian of $\lambda f(x) + \mu\phi(x) \equiv \kappa\{\lambda f(x) + \mu\phi(x)\}$, from which identity we have five equations to eliminate $\lambda^2, \lambda\mu, \mu^2, \kappa\lambda, \kappa\mu$; thus obtaining an invariant I_{44}, of the 4^{th} degree in the coefficients of each equation.

15. The discriminant of $\lambda U + \mu V$, where U and V are cubics $(a, b, c, d)(x, y)^3$, $(a', b', c', d')(x, y)^3$, being written as in Art. 192, resolve into its factors the covariant

$$(\Delta, \Theta, \Phi, \Theta', \Delta')(V, -U)^4.$$

The leading coefficient of this covariant is easily obtained by forming the discriminant of $aV - a'U$ directly; it is

$$(ab')^2\{4(ab')(ad') - 3(ac')^2\},$$

which may be written in the form $2A^2\{PA + 6(AC - B^2)\}$, where A, B, C are the first three coefficients of the Jacobian; and, consequently, the given covariant is expressed as follows :—

$$2J^2(U, V)\{PJ(U, V) + 6 \text{ Hessian of } J(U, V)\}.$$

16. Express the invariants of the Jacobian of two cubics in terms of P and Q.

Ans. $12I' = P^2, \quad 216J' = 54Q - P^3.$

CHAPTER XIX.

TRANSFORMATIONS.

SECTION I.—TSCHIRNHAUSEN'S TRANSFORMATIONS.

194. Under the general heading of this chapter we propose collecting several propositions which could not have been conveniently given elsewhere, and which are of importance in connexion with the subjects discussed in the foregoing pages. We commence with a general theorem relating to rational transformations.

Theorem.—*The most general rational algebraic transformation of a root of an equation of the n^{th} degree can be reduced to an integral transformation of the degree $n-1$ at most.*

For every rational function of a root α_r of the equation

$f(x)=0$ is of the form

$$\frac{\chi(\alpha_r)}{\psi(\alpha_r)},$$

where χ and ψ are integral functions; also,

$$\frac{\chi(\alpha_r)}{\psi(\alpha_r)}=\chi(\alpha_r)\frac{\psi(\alpha_1)\ldots\ldots\psi(\alpha_{r-1})\,\psi(\alpha_{r+1})\ldots\ldots\psi(\alpha_n)}{\psi(\alpha_1)\,\psi(\alpha_2)\ldots\ldots\ldots\ldots\psi(\alpha_{n-1})\,\psi(\alpha_n)},$$

and the denominator $\psi(\alpha_1)\,\psi(\alpha_2)\ldots\psi(\alpha_n)$, being a symmetric function of the roots of $f(x)=0$, can be expressed as a rational function of the coefficients. Whence $\dfrac{\chi(\alpha_r)}{\psi(\alpha_r)}$ is reduced to an integral form.

Moreover, the numerator of the former fraction is a symmetric function of the roots of the equation $\dfrac{f(x)}{x-\alpha_r}=0$, and may consequently be expressed as a rational function of the coefficients of that equation; that is, in terms of α_r and the coefficients of $f(x)$.

Now, denoting by $F(a_r)$ this integral form of $\frac{\chi(a_r)}{\psi(a_r)}$, we have by division

$$F(a_r) - Qf(a_r) + \phi(a_r) = \phi(a_r),$$

where $\phi(a_r)$ does not exceed the degree $n-1$; which proves the proposition.

In the particular cases of the quadratic and cubic it follows that the most general rational function of a root can be reduced to a linear function, and a quadratic function of that root, respectively. In the case of the cubic this quadratic function may be reduced to another form which is often useful, as follows: Denoting the quadratic function by $\psi(\theta)$, and dividing the cubic $f(\theta)$ by $\psi(\theta)$, we have

$$f(\theta) = (q_0 + q_1\theta)\,\psi(\theta) + r_0 + r_1\theta = 0,$$

proving that

$$\psi(\theta) = -\frac{r_0 + r_1\theta}{q_0 + q_1\theta};$$

whence it appears that *the most general transformation of a root of a cubic may be reduced to a homographic transformation.*

In connexion with the proposition here established it is easy to justify the remarks made in Arts. 59, 66, relative to the solutions of the cubic and the biquadratic equations. With this object in view, let ϕ and ψ be two rational functions of n quantities $a_1, a_2, \ldots a_n$ (which may be considered as the roots of an equation), each having only p values when the roots are interchanged in every way. Denoting these values of both functions obtained by the same substitutions by

$$\phi_1, \phi_2, \phi_3, \ldots \phi_p,$$
$$\psi_1, \psi_2, \psi_3, \ldots \psi_p,$$

we have, for every integer j,

$$\phi_1\psi_1{}^j + \phi_2\psi_2{}^j + \phi_3\psi_3{}^j + \ldots + \phi_p\psi_p{}^j = T_j;$$

a symmetric function of the roots, since it is the sum of all the possible values which $\phi\psi^j$ can take.

In this way we may obtain the system of equations

$$\begin{array}{lllll} \phi_1 & + \phi_2 & + \phi_3 & + \ldots \phi_p & = T_0, \\ \phi_1\psi_1 & + \phi_2\psi_2 & + \phi_3\psi_3 & + \ldots \phi_p\psi_p & = T_1, \\ . & . & . & . & . \\ \phi_1\psi_1^{p-1} & + \phi_2\psi_2^{p-1} & + \phi_3\psi_3^{p-1} & + \ldots \phi_p\psi_p^{p-1} & = T_{p-1}, \end{array}$$

where T_0, $T_1, \ldots T_{p-1}$ are all symmetric functions of $a_1, a_2, a_3, \ldots a_n$.

Solving these equations, we find at once ϕ_1 expressed as a symmetric function of $\psi_2, \psi_3, \ldots \psi_p$, since any interchange of $\psi_2, \psi_3, \ldots \psi_p$, being equivalent to an interchange of $\phi_2, \phi_3, \ldots \phi_p$, does not alter the value of ϕ_1. This value, therefore, is by the present proposition reducible to a rational and integral function of ψ_1 of the degree $p - 1$, since ψ has only p values considered as a function of $a_1, a_2, \ldots a_n$. Now considering the special cases referred to—(1) when $p = 2$, and $n = 3$, it is proved that a linear relation connects ϕ and ψ in terms of symmetric functions of a_1, a_2, a_3; and (2), when $p = 3$, and $n = 4$, ϕ and ψ may be shown to be connected by a rational relation (see Examples 5, 6, 7, p. 132, Vol. I.; Ex. 3, p. 106, Vol. II.).

195. **Formation of the Transformed Equation.**—The transformation explained in the preceding Article was first employed by Tschirnhausen for the reduction of the cubic and biquadratic. We may form in general the equation whose roots are $\phi(a_1), \phi(a_2), \ldots \phi(a_n)$, where

$$\phi(x) \equiv a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1}x^{n-1}.$$

is an integral function of x of the degree $n - 1$, by putting $\phi(x) = y$, and eliminating x from the equations

$$f(x) = 0, \quad y = a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1}x^{n-1};$$

or we may proceed by raising $\phi(x)$ to the different powers $2, 3, \ldots n$ in succession, and reducing the exponents of x in each case below n (by dividing by $f(x)$ and retaining only the remainder), we have

$$\phi^2 = b_0 + b_1 x + b_2 x^2 + \ldots + b_{n-1}x^{n-1},$$

$$\phi^3 = c_0 + c_1 x + c_2 x^2 + \ldots + c_{n-1}x^{n-1},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\phi^n = l_0 + l_1 x + l_2 x^2 + \ldots + l_{n-1}x^{n-1}.$$

Substituting for x in these equations each of the roots of the equation $f(x) = 0$, and adding, we find, if S_1, S_2, S_3, &c., denote the sums of the powers of the roots of the required equation,

$$S_1 = na_0 + a_1 s_1 + a_2 s_2 + \ldots + a_{n-1}s_{n-1},$$

$$S_2 = nb_0 + b_1 s_1 + b_2 s_2 + \ldots + b_{n-1}s_{n-1},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$S_n = nl_0 + l_1 s_1 + l_2 s_2 + \ldots + l_{n-1}s_{n-1}.$$

Now, expressing s_1, s_2, ... s_{n-1} in terms of the coefficients of $f(x)$, we have S_1, S_2, ... S_n determined in terms of the coefficients of $\phi(x)$ and $f(x)$; we are also enabled by Art. 80 to express the coefficients of the equations whose roots are $\phi(\alpha_1)$, $\phi(\alpha_2)$, ... $\phi(\alpha_n)$ in terms of S_1, S_2, ... S_n, and therefore finally in terms of the coefficients of $\phi(x)$ and $f(x)$; thus theoretically the transformation is completed.

196. **Another Method of forming the Transformed Equation.**—There is another way of finding the final equation in ϕ by elimination, which we now give. Since

$$a_0 - \phi + a_1 x + a_2 x^2 + \ldots - a_{n-1}x^{n-1} = 0,$$

if this equation be multiplied by x, x^2, ... x^{n-1}, and the exponents of x reduced below n by means of the equation $f(x) = 0$, we have in all n equations to eliminate dialytically the $n - 1$ quantities, x, x^2, ... x^{n-1}. We thus obtain the transformed equation in the form of a determinant of the n^{th} order, ϕ entering into the diagonal constituents only. For example, if $f(x) = x^n - 1$, we obtain the transformed equation in the following form :—

$$\begin{vmatrix} a_0 - \phi & a_1 & a_2 & . & . & a_{n-1} \\ a_{n-1} & a_0 - \phi & a_1 & . & . & a_{n-2} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_1 & a_2 & a_3 & . & . & a_0 - \phi \end{vmatrix} = 0.$$

Although these methods of performing Tschirnhausen's transformation appear simple, yet if they be applied to particular cases, the result usually appears in a complicated form. Professor Cayley, by choosing a form of the transformation suggested by M. Hermite, was enabled to take advantage of the theory of covariants, and thus to complete the transformation for the cubic, quartic, and quintic. We shall content ourselves with showing in an elementary way how Cayley's results for the cubic and quartic may be obtained.

197. **Tschirnhausen's Transformation applied to the Cubic.**—Let the cubic equation

$$ax^3 + 3bx^2 + 3cx + d = 0$$

be written under the form

$$z^3 + 3Hz + G = 0;$$

and let it be transformed by the substitution

$$y = \lambda + \kappa z + z^2.$$

If z_1, z_2, z_3 be the roots of the cubic, and y_1, y_2, y_3 the corresponding values of y, we have

$$\left.\begin{aligned} y_2 - y_3 &= (z_2 - z_3)(\kappa - z_1), \\ y_3 - y_1 &= (z_3 - z_1)(\kappa - z_2), \\ y_1 - y_2 &= (z_1 - z_2)(\kappa - z_3), \end{aligned}\right\} \quad (1)$$

and consequently

$$\left.\begin{aligned} 2y_1 - y_2 - y_3 &= (2z_1 - z_2 - z_3)\,\kappa + (2z_2z_3 - z_3z_1 - z_1z_2), \\ 2y_2 - y_3 - y_1 &= (2z_2 - z_3 - z_1)\,\kappa + (2z_3z_1 - z_1z_2 - z_2z_3), \\ 2y_3 - y_1 - y_2 &= (2z_3 - z_1 - z_2)\,\kappa + (2z_1z_2 - z_2z_3 - z_3z_1). \end{aligned}\right\} \quad (2)$$

Wherefore, if the equation in y with the second term removed be

$$Y^3 + 3H'Y + G' = 0,$$

we have from equations (1) and (2)

$$H' = H_\kappa, \quad G' = G_\kappa,$$

where H_κ and G_κ are the Hessian and cubic covariant of

$$\kappa^3 + 3H\kappa + G;$$

and the transformation is therefore completed, since $y_1 + y_2 + y_3$ can be easily determined.

198. **The Tschirnhausen Transformation applied to the Quartic.**—In this case we do not attempt to form directly the transformed quartic, but prove the following theorem, which shows how this transformation may be resolved into two others.

Theorem.—*The Tschirnhausen transformation changes a quartic U into one having the same invariants as $lU + mH_x$, and therefore in general reducible to the latter form by linear transformation.*

To prove this, let the quartic

$$x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = 0$$

be transformed by the substitution of the most general expression for a root of a quartic.

$$y = a_0 + a_1x + a_2x^2 + a_3x^3.$$

If x_1, x_2, x_3, x_4 be the roots of the quartic, and y_1, y_2, y_3, y_4 the corresponding values of y, we have

$$\frac{y_2 - y_3}{x_2 - x_3} = a_1 + a_2(x_2 + x_3) + a_3(x_2^2 + x_2x_3 + x_3^2),$$

$$\frac{y_1 - y_4}{x_1 - x_4} = a_1 + a_2(x_1 + x_4) + a_3(x_1^2 + x_1x_4 + x_4^2).$$

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From these equations we proceed to show that

$$\frac{(y_2 - y_3)(y_1 - y_4)}{(x_2 - x_3)(x_1 - x_4)} = P_0 + Q_0(x_2x_3 + x_1x_4),$$

where P_0 and Q_0 involve the roots of the quartic symmetrically.

In the first place, we find

$$(x_2^2 + x_2x_3 + x_3^2)(x_1^2 + x_1x_4 + x_4^2) = p_2^2 - p_1p_3 + p_4 - p_2\lambda,$$

where λ has its usual value, viz., $x_2x_3 + x_1x_4$; and, secondly, since

$$x_2^2 + x_2x_3 + x_3^2 = (x_2 + x_3)^2 - x_2x_3, \text{ \&c.},$$

we find again

$$(x_2+x_3)(x_1^2+x_1x_4+x_4^2)+(x_1+x_4)(x_2^2+x_2x_3+x_3^2)=p_3-p_1p_2+p_1\lambda.$$

Finally, since the other terms in the product are obviously of the same form as $P_0 + Q_0\lambda$, we have proved that

$$\frac{(y_2 - y_3)(y_1 - y_4)}{(x_2 - x_3)(x_1 - x_4)} = P_0 + Q_0(x_2x_3 + x_1x_4);$$

whence

$$(y_2 - y_3)(y_1 - y_4) = (\nu - \mu)(P_0 + Q_0\lambda).$$

Now, introducing ρ_1, ρ_2, ρ_3 in place of λ, μ, ν this and the similar equations preserve their forms; whence, altering P_0 and Q_0 into similar quantities, we obtain the equations

$$(y_2 - y_3)(y_1 - y_4) = 4(\rho_3 - \rho_2)(P - Q\rho_1),$$
$$(y_3 - y_1)(y_2 - y_4) = 4(\rho_1 - \rho_3)(P - Q\rho_2),$$
$$(y_1 - y_2)(y_3 - y_4) = 4(\rho_2 - \rho_1)(P - Q\rho_3),$$

which lead at once to the invariants of the transformed quartic; and comparing their values with the invariants of $\kappa U - \lambda H_x$ given in Art. 187, the theorem follows at once.

199. Reduction of the Cubic to a Binomial form by the Tschirnhausen Transformation.—Let the cubic

$$ax^3 + 3bx^2 + 3cx + d$$

be reduced to the form $y^3 - V$ by the transformation

$$y = q + px + x^2.$$

If x_1, x_2, x_3 be the roots of the given cubic, and y_1 a root of the transformed cubic, we have the following equations to determine p and q :—

$$x_1^2 + px_1 + q = y_1,$$
$$x_2^2 + px_2 + q = \omega y_1,$$
$$x_3^2 + px_3 + q = \omega^2 y_1;$$

from which we find

$$p = -\frac{x_1^2 + \omega x_2^2 + \omega^2 x_3^2}{x_1 + \omega x_2 + \omega^2 x_3}, \quad q = -\tfrac{1}{3}(s_2 + ps_1).$$

Adding $x_1 + x_2 + x_3$ to this value of p, we have

$$p + x_1 + x_2 + x_3 = -\frac{x_2 x_3 + \omega x_3 x_1 + \omega^2 x_1 x_2}{x_1 + \omega x_2 + \omega^2 x_3};$$

it follows (Ex. 25, p. 57, Vol. I.) that there are only two ways of completing this transformation, as the values of p, q ultimately depend on the solution of the Hessian of the cubic.

200. **Reduction of the Quartic to a Trinomial Form by Tschirnhausen's Transformation.**—Let the quartic

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e$$

be reduced to the form $y^4 + Py^2 + Q$, in which the second and fourth terms are absent, by the transformation

$$y = q + px + x^2.$$

If x_1, x_2, x_3, x_4 be the roots of the quartic; also y_1, y_2 two *distinct* roots of the transformed quartic, we have the following equations to determine p and q :—

$$x_1^2 + px_1 + q = y_1, \quad x_3^2 + px_3 + q = y_2,$$
$$x_2^2 + px_2 + q = -y_1, \quad x_4^2 + px_4 + q = -y_2;$$

from which we find

$$p = -\frac{x_1^2 + x_2^2 - x_3^2 - x_4^2}{x_1 + x_2 - x_3 - x_4}, \quad q = -\tfrac{1}{4}(s_2 + ps_1).$$

And, adding $x_1 + x_2 + x_3 + x_4$ to this value of p, we have

$$p + x_1 + x_2 + x_3 + x_4 = \frac{2(x_1 x_2 - x_3 x_4)}{x_1 + x_2 - x_3 - x_4};$$

hence, by Ex. 5, p. 132, Vol. I., it follows that there are three ways of reducing the quartic to the proposed form, the determination of which ultimately depends on the solution of the reducing cubic of the quartic.

201. **Removal of the Second, Third and Fourth Terms from an Equation of the n^{th} Degree.**—We begin by proving the following proposition, which we shall subsequently apply :—

A homogeneous function V of the second degree in n variables $x_1, x_2, x_3, \ldots x_n$ can be expressed as the sum of n squares.

To prove this, let V arranged in powers of x_1 take the form

$$V \equiv p_0 x_1^2 + 2p_1 x_1 + p_2$$

where p_0 is a constant, p_1 a linear function, and p_2 a quadratic function of $x_2, x_3, \ldots x_n$.

(*a*) If p_0 does not vanish, $V \equiv p_0\left(x_1 + \dfrac{p_1}{p_0}\right)^2 + p_2 - \dfrac{p_1^2}{p_0}$, and as $p_2 - \dfrac{p_1^2}{p_0}$ does not contain x_1, we have reduced the question to one of expressing a homogeneous function of the second degree in $n-1$ variables as a sum of $n-1$ squares with constant coefficients.

(*b*) If $p_0 = 0$, and we wish to deal with x_1, the coefficient of the product of x_1 and some other variable, say x_2, must not vanish, otherwise V would be independent of x_1. Write, then, V in the form $ax_1x_2 + cx_2^2 + p_1x_1 + q_1x_2 + r_2$, where a, c are constants, p_1, q_1 linear functions, and r_2 a quadratic function of $x_3, x_4, \ldots x_n$. Therefore,

$$V \equiv x_2(ax_1 + cx_2) + (ax_1 + cx_2)\frac{p_1}{a} + \left(q_1 - \frac{cp_1}{a}\right)x_2 + r_2$$

$$\equiv \left(x_2 + \frac{p_1}{a}\right)\left(ax_1 + cx_2 + q_1 - \frac{cp_1}{a}\right) - \frac{p_1}{a}\left(q_1 - \frac{cp_1}{a}\right) + r_2$$

$$\equiv ay_1y_2 + r_2'$$

where $$y_1 = x_1 + \frac{c}{a}x_2 + \frac{q_1}{a} - \frac{cp_1}{a^2}, \qquad y_2 = x_2 + \frac{p_1}{a},$$

and r_2' is a quadratic function of $x_3, x_4, \ldots x_n$.

$$\therefore V \equiv \frac{a}{2}(z_1^2 - z_2^2) + r_2'$$

where $$z_1 = \frac{y_1 + y_2}{\sqrt{2}}, \quad z_2 = \frac{y_2 - y_1}{\sqrt{2}}.$$

We have thus reduced the question to one of expressing a homogeneous function of the second degree in $n - 2$ variables as a sum of $n - 2$ squares with constant coefficients.

Note if $x_3, x_4, \ldots x_n$ are put equal to zero, we have arranged the modulus of transformation between x_1, x_2, and z_1, z_2, so that it is equal to unity.

Proceeding then by (a) or (b) as required, we finally express V as the sum of n squares, for any one of the squares, say aX^2, may be written $(\sqrt{a}X)^2$.

Now, returning to the original problem, let the equation be

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0;$$

and, putting

$$y = \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \epsilon,$$

let the transformed equation be

$$y^n + Q_1 y^{n-1} + Q_2 y^{n-2} + \ldots + Q_n = 0,$$

where, by Art. 195, $Q_1, Q_2, \ldots Q_r, \ldots$ are homogeneous functions of the first, second, $\ldots r^{th}$ degrees in $\alpha, \beta, \gamma, \delta, \epsilon$.

Now, if $\alpha, \beta, \gamma, \delta, \epsilon$ can be determined so that

$$Q_1 = 0, \quad Q_2 = 0, \quad Q_3 = 0,$$

the problem will be solved. For this purpose, eliminating ϵ from Q_2 and Q_3, by substituting its value derived from $Q_1 = 0$, we obtain two homogeneous equations

$$R_2 = 0, \quad R_3 = 0,$$

of the second and third degrees in $\alpha, \beta, \gamma, \delta$; and by the proposition proved above, we may write R_2 under the form

$$u^2 - v^2 + w^2 - t^2,$$

which is satisfied by putting $u = v$ and $w = t$. From these simple equations we find $\gamma = l\alpha + m\beta$, and $\delta = l_1\alpha + m_1\beta$; and substituting these values in $Q_3 = 0$, we have a cubic equation to determine the ratio $\beta : \alpha$. Whence, giving any one of the quantities α, β, γ, δ, ϵ a definite value, the rest are determined, and the equation is reduced to the form

$$y^n + Q_4 y^{n-4} + Q_5 y^{n-5} + \ . \ . \ . + Q_n = 0.$$

In a similar way we may remove the coefficients Q_1, Q_2, Q_4, by solving an equation of the fourth degree.

Applying this method to the quintic, we may reduce it to either of the trinomial forms

$$x^5 + Px + Q, \quad x^5 + Px^2 + Q;$$

or again, changing x into $\frac{1}{x}$, to either of the forms

$$x^5 + Px^3 + Q, \quad x^5 + Px^4 + Q.$$

In this investigation we have followed M. Serret (see his *Cours d'Algèbre Supérieure*, Vol. I., Art. 192).

Section II.—Hermite's and Sylvester's Theorems.

202. **Homogeneous Function of Second Degree expressed as Sum of Squares.**—We have already shown, (Art. 201) that a homogeneous function of the second degree in the variables may be reduced to a sum of squares, no hypothesis being made as to the nature of the coefficients of the function considered. We now return to the consideration of this problem when the coefficients of the function are supposed to be all *real*; and we proceed to determine, in magnitude and sign, the coefficients of the squares in the transformed function.

Let $F(x_1, x_2, \ . \ . \ . \ x_n)$ be a homogeneous function of the second degree in n variables, with real coefficients; and let us suppose that it is reduced, using the method (a) of Art. 201 alone, to the form

$$\begin{aligned} p_1(x_1 + a_2x_2 + a_3x_3 + \ . \ . \ . + a_nx_n)^2 & \\ + p_2(x_2 + b_3x_3 + \ . \ . \ . + b_nx_n)^2 & \\ + p_3(x_3 + \ . \ . \ . + c_nx_n)^2 & \\ . \ . \ . \ . \ . \ . \ . \ . \ . \ . & \\ + p_nx_n^2, & \end{aligned}$$

where all the coefficients of this new form are real.

Making now the linear transformation

$$\begin{aligned}
X_1 &= x_1 + a_2x_2 + a_3x_3 + a_4x_4 + \dots + a_nx_n, \\
X_2 &= \qquad\quad x_2 + b_3x_3 + b_4x_4 + \dots + b_nx_n, \\
X_3 &= \qquad\qquad\qquad x_3 + c_4x_4 + \dots + c_nx_n, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
X_n &= \qquad\qquad\qquad\qquad\qquad\qquad l_nx_n,
\end{aligned}$$

we have

$$F(x_1, x_2, x_3, \dots x_n) = p_1X_1^2 + p_2X_2^2 + p_3X_3^2 + \dots + p_nX_n^2.$$

Since the modulus of this transformation is equal to 1, the discriminants of both these forms of F must be absolutely equal. Denoting, therefore, the discriminant of F by Δ_n, we have

$$\Delta_n = p_1p_2p_3 \dots p_n;$$

and similarly, when the variables $x_{j+1}, x_{j+2}, \dots x_n$ are made to vanish in both forms of F, we have

$$\Delta_j = p_1p_2p_3 \dots p_j.$$

Now, giving j the values 1, 2, 3, &c., we find, assuming $p_1 = 4$,

$$p_1 = \Delta_1, \; p_2 = \frac{\Delta_2}{\Delta_1}, \; p_3 = \frac{\Delta_3}{\Delta_2}, \; \dots \; p_n = \frac{\Delta_n}{\Delta_{n-1}},$$

and the coefficients are determined in terms of the discriminants of the original quadratic form in n variables and the discriminants of the forms in $n-1$, $n-2$, &c., variables derived from the given form by causing one, two, &c., of the variables to vanish in succession in the manner just explained.

If we have to use the method (*b*) of Art. 201 in expressing F in the form $F \equiv p_1X_1^2 + p_2X_2^2 + \dots + p_nX_n^2$, we note that whenever we do so, say in dealing with x_3, x_4, we have

$$\begin{aligned}
\sqrt{2}X_3 &= x_3 + (1 + c)\,x_4 + a_5x_5 + \dots + a_nx_n, \\
\sqrt{2}X_4 &= -x_3 + (1 - c)\,x_4 + b_5x_5 + \dots + b_nx'_n,
\end{aligned}$$

and that $p_4 = -p_3$. We see then that the moduli of transformation are still all equal to 1. When, however, we make

$x_4, x_5 \ldots x_n$ vanish, $X_3^2 = X_4^2$, so that $p_3X_3^2 + p_4X_4^2 \equiv 0$, and $\Delta_3 = 0$, but $\Delta_4 = -p_1p_2p_3^2$, and $\Delta_2 = p_1p_2$, $\therefore p_3^2 = -\Delta_4/\Delta_2$. Generally, then, when $\Delta_r = 0$, $p_r = -p_{r+1}$, and $p_r^2 = -\Delta_{r+1}/\Delta_{r-1}$. By this method we determine p_r, p_{r+1} in absolute magnitude, but not in sign. It is important also to note that if Δ_r vanishes, Δ_{r+1} and Δ_{r-1} have opposite sign.

Again, although F can be reduced to a sum of squares in a great number of ways, it is most important to observe that *in whatever way the transformation is made, provided it is real, the number of coefficients (affecting these squares) which have a given sign is always the same.* This theorem, which is due to Jacobi, is easily proved; for suppose the contrary possible, and let

$$F \equiv p_1X_1^2 + p_2X_2^2 + \ldots + p_nX_n^2 \equiv q_1Y_1^2 + q_2Y_2^2 + \ldots + q_nY_n^2,$$

where the number of positive coefficients on both sides of this identity is not the same. Making all the terms positive, by transferring those affected with negative signs to the opposite sides of the identity, we shall have a sum of l squares identically equal to a sum of m squares, where m is greater than l. Now, substituting such values for $x_1, x_2, \ldots x_n$ that each of the l squares may vanish (which may be done in an infinity of ways), we find a sum of m squares identically equal to zero, which is impossible.

203. **Hermite's Theorem.**—The principles explained in the preceding Article have been applied by Hermite to the determination of the number of real roots of the equation $f(x) = 0$ comprised within given limits. The special form of the equation F which he makes use of for this purpose is

$$\sum_{r=1}^{r=n} \frac{1}{\alpha_r - \rho}(x_1 + \alpha_r x_2 + \alpha_r^2 x_3 + \ldots + \alpha_r^{n-1}x_n)^2,$$

in which $x_1, x_2, \ldots x_n$ are any variables in number equal to the degree of the equation; and r takes all values from 1 to n inclusive, the roots of the equation being $\alpha_1, \alpha_2, \ldots \alpha_n$; also ρ is an arbitrary parameter.

This form is plainly a symmetric function of the roots of the equation $f(x) = 0$; and as the coefficients of this equation are supposed to be real, F will be also real, when expressed in terms of these coefficients and ρ, provided the parameter ρ be given any real value. If the roots $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ are not all real, the assumed form of F will not be obtained by a real transformation; but it is easy to deduce from it, as follows, another form which will be so obtained.

If α_1 and α_2 be a pair of conjugate imaginary roots, we may write

$$\alpha_1 = r_0 (\cos\alpha + i\sin\alpha), \quad \alpha_2 = r_0 (\cos\alpha - i\sin\alpha).$$

Denoting for shortness $x_1 + \alpha_r x_2 + \alpha_r^2 x_3 + \ldots + \alpha_r^{n-1} x_n$ by Y_r, and substituting these values in Y_1 and Y_2, we find

$$Y_1 \equiv U + iV, \quad Y_2 \equiv U - iV,$$

where U and V are real; also putting

$$\frac{1}{\alpha_1 - \rho} = r(\cos\phi + i\sin\phi), \quad \frac{1}{\alpha_2 - \rho} = r(\cos\phi - i\sin\phi),$$

the part of the function F depending on α_1 and α_2, viz.,

$$\frac{Y_1^2}{\alpha_1 - \rho} + \frac{Y_2^2}{\alpha_2 - \rho},$$

becomes

$$r\left\{\left(\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right)^2 (U + iV)^2 + \left(\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\right)^2 (U - iV)^2\right\},$$

which may be also written as the difference of the squares

$$2r\left(U\cos\frac{\phi}{2} - V\sin\frac{\phi}{2}\right)^2 - 2r\left(U\sin\frac{\phi}{2} + V\cos\frac{\phi}{2}\right)^2;$$

proving that, if ρ is real, two imaginary conjugate roots introduce into F two real squares, one of which has a positive and the other a negative coefficient.

We now state Hermite's theorem as follows: *Let the equation*

$f(x) \equiv (x-a_1)(x-a_2)\ldots(x-a_n)=0$ have real coefficients and unequal roots: if then by a REAL *substitution we reduce*

$$\frac{Y_1^2}{a_1-\rho}+\frac{Y_2^2}{a_2-\rho}+\frac{Y_3^2}{a_3-\rho}+\ldots+\frac{Y_n^2}{a_n-\rho}, \tag{1}$$

where $\quad Y_r = x_1 + a_r x_2 + a_r^2 x_3 + \ldots + a_r^{n-1} x_n,$

to a sum of squares, the number of squares having positive coefficients will be equal to the number of pairs of imaginary roots of the equation $f(x)=0$, augmented by the number of real roots greater than ρ.

The theorem will be also true if $(a_r-\rho)^m$ is substituted for $a_r-\rho$, where m is any odd integer, positive or negative.

The theorem follows at once from what has preceded if we consider separately the parts of the function (1) which refer to real roots and to imaginary roots, for obviously there is a positive square for every root greater than ρ; and we have proved that every pair of conjugate imaginary roots leads to a positive and negative real square, without affecting the other squares independent of these roots.

The number of real roots between any two numbers ρ_1 and ρ_2 may be readily estimated. For, denoting in general by P_j the number of positive squares in F when $\rho=\rho_j$, by N_j the number of roots of the equation $f(x)=0$ greater than ρ_j, and by $2I$ the number of imaginary roots, we have

$$P_1 = N_1 + I, \quad P_2 = N_2 + I;$$

whence $\quad N_1 - N_2 = P_1 - P_2,$

proving that the number of real roots between ρ_1 and ρ_2 is equal to the difference between the number of positive or negative squares when ρ has the values ρ_1 and ρ_2.

The number here determined may be shown to depend on a very important series of functions connected with the given equation. In order to derive these functions, we consider F under the form (a), Art. 202:—

$$\Delta_1 X_1^2 + \frac{\Delta_2}{\Delta_1}X_2^2 + \frac{\Delta_3}{\Delta_2}X_3^2 + \ldots + \frac{\Delta_n}{\Delta_{n-1}}X_n^2,$$

in which it can be expressed if none of $\Delta_1, \Delta_2, \ldots \Delta_n$ vanish.

The number P expresses the number of coefficients in this form which are positive, or, which is the same thing, the number of the following quantities which are negative :—

$$-\frac{\Delta_1}{1}, -\frac{\Delta_2}{\Delta_1}, -\frac{\Delta_3}{\Delta_2}, \ldots -\frac{\Delta_n}{\Delta_{n-1}}. \qquad (2)$$

We proceed now to calculate $\Delta_1, \Delta_2, \ldots \Delta_j, \ldots \Delta_n$ in terms of ρ and the roots of the equation $f(x) = 0$; and as the method is similar in every case, it will be sufficient to calculate Δ_3, i.e. the discriminant of the original form of F when all the variables except x_1, x_2, x_3 vanish.

Writing for shortness $\nu_r = \dfrac{1}{\alpha_r - \rho}$, we have in this case

$$F_3 = \Sigma\nu_r (x_1 + \alpha_r x_2 + \alpha_r^2 x_3)^2.$$

The discriminant in this form is

$$\Delta_3 = \begin{vmatrix} \Sigma\nu & \Sigma\alpha\nu & \Sigma\alpha^2\nu \\ \Sigma\alpha\nu & \Sigma\alpha^2\nu & \Sigma\alpha^3\nu \\ \Sigma\alpha^2\nu & \Sigma\alpha^3\nu & \Sigma\alpha^4\nu \end{vmatrix},$$

which may be written as the product of the two arrays

$$\left.\begin{matrix} 1 & 1 & \ldots & 1 \\ \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & \ldots & \alpha_n^2 \end{matrix}\right\}, \quad \left.\begin{matrix} \nu_1 & \nu_2 & \ldots & \nu_n \\ \alpha_1\nu_1 & \alpha_2\nu_2 & \ldots & \alpha_n\nu_n \\ \alpha_1^2\nu_1 & \alpha_2^2\nu_2 & \ldots & \alpha_n^2\nu_n \end{matrix}\right\};$$

and consequently

$$\Delta_3 = \Sigma\nu_1\nu_2\nu_3 \begin{vmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{vmatrix}^2 = \sum \frac{(\alpha_2 - \alpha_3)^2 (\alpha_3 - \alpha_1)^2 (\alpha_1 - \alpha_2)^2}{(\alpha_1 - \rho)(\alpha_2 - \rho)(\alpha_3 - \rho)}.$$

In a similar manner we find

$$\Delta_j = \sum \frac{\nabla(\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_j)}{(\alpha_1 - \rho)(\alpha_2 - \rho) \ldots (\alpha_j - \rho)},$$

where the notation $\nabla(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_j)$ is employed to represent the product of the squares of the differences of $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_j$. Hence the quantities $\Delta_1, \Delta_2, \ldots \Delta_j, \ldots \Delta_n$ are all determined.

Now, multiplying the numerator and denominator of each of the fractions in the series (2) by $f(\rho)$, each value of Δ is rendered integral, and the series becomes

$$\frac{V_1}{V}, \quad \frac{V_2}{V_1}, \quad \frac{V_3}{V_2}, \quad \ldots \quad \frac{V_n}{V_{n-1}}, \qquad (3)$$

where

$$\begin{aligned}
V &= (\rho - \alpha_1)(\rho - \alpha_2) \ldots (\rho - \alpha_n), \\
V_1 &= \Sigma (\rho - \alpha_2)(\rho - \alpha_3) \ldots (\rho - \alpha_n), \\
V_2 &= \Sigma\nabla(\alpha_1, \alpha_2)(\rho - \alpha_3) \ldots (\rho - \alpha_n), \\
V_3 &= \Sigma\nabla(\alpha_1, \alpha_2, \alpha_3)(\rho - \alpha_4) \ldots (\rho - \alpha_n), \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
V_n &= \Delta(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n).
\end{aligned}$$

Since negative terms in the series (3) correspond to variations of sign in the series $V, V_1, V_2, \ldots V_n$, it is proved that the number of variations lost in the series last written, when ρ passes from the value ρ_1 to the value ρ_2, is exactly equal to the number of real roots of the equation $f(\rho) = 0$ comprised between ρ_1 and ρ_2.

As $V, V_1, V_2, \ldots V_n$ are derived from $1, \Delta_1, \Delta_2, \ldots \Delta_n$, by multiplying the latter by $f(\rho)$, and as when $\Delta_r = 0$, Δ_{r+1} and Δ_{r-1} have opposite sign, we note that when V_r vanishes for a value of ρ not equal to a root of $f(\rho)$, V_{r+1} and V_{r-1} have opposite sign.

It will be observed that the functions V, V_1, V_2, &c., here arrived at have the same properties as Sturm's functions; from which in fact they differ by positive multipliers only, as was observed by Sylvester, who first published these forms in the *Philosophical Magazine*, December, 1839. In order to establish the identity of the two series of functions, we proceed in the first place to prove in the following Article an important theorem connecting the leading coefficients of Sturm's functions and the sums of the powers of the roots of an equation.

204. **Theorem.**—*The leading coefficients of Sturm's auxiliary*

functions (i.e. $f'(x)$, *and the* $n-1$ *remainders*) *differ by positive factors only from the following series of determinants* :—

$$s_0, \quad \begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix}, \quad \begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix}, \quad \begin{vmatrix} s_0 & s_1 & s_2 & s_3 \\ s_1 & s_2 & s_3 & s_4 \\ s_2 & s_3 & s_4 & s_5 \\ s_3 & s_4 & s_5 & s_6 \end{vmatrix}, \text{ \&c.}$$

Using the bracket notation, we may write these determinants in the form s_0, $(s_0 s_2)$, (s_0, s_2, s_4), &c., the last in the series being $(s_0 s_2 s_4 \ . \ . \ . \ s_{2n-2})$.

Representing Sturm's remainders by $R_2, R_3, \ldots R_j, \ldots R_n$, and the successive quotients by Q_1, Q_2, Q_3, &c., we have (see Art. 96)

$$R_2 = Q_1 f'(x) - f(x),$$
$$R_3 = Q_2 R_2 - f'(x) = (Q_1 Q_2 - 1) f'(x) - Q_2 f(x),$$
$$R_4 = Q_3 R_3 - R_2 = (Q_1 Q_2 Q_3 - Q_1 - Q_3) f'(x) - (Q_2 Q_3 - 1)(fx), \text{ \&c.}$$

Proceeding in this manner, we observe that any remainder R_j can be expressed in the form

$$\mathrm{R}_j \equiv A_j f'(x) - B_j f(x). \qquad (1)$$

The degree of R_j is $n - j$; and since Q_1, Q_2, &c., are all of the first degree in x, it appears that the degrees of A_j and B_j are $j-1$ and $j-2$, respectively.

Assuming therefore, for R_j and A_j, the forms

$$R_j \equiv r_0 + r_1 x + r_2 x^2 + \ . \ . \ . + r_{n-j} x^{n-j},$$
$$A_j \equiv \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \ . \ . \ . + \lambda_{j-1} x^{j-1},$$

and substituting in (1) any root α of the equation $f(x) = 0$, we have

$$\lambda_0 + \lambda_1 \alpha + \lambda_2 \alpha^2 + \ . \ . \ . + \lambda_{j-1} \alpha^{j-1} = \frac{r_0 + r_1 \alpha + r_2 \alpha^2 + \ . \ . \ . + r_{n-j} \alpha^{n-j}}{f'(\alpha)}.$$

Multiplying by $\alpha, \alpha^2, \ldots \alpha^{j-2}, \alpha^{j-1}$ in succession; making similar substitutions of the other roots; and adding the equations thus derived, we obtain, by aid of the relations of Ex. 4, p. 172, Vol. I., the following system of equations :—

$$\begin{aligned}
\lambda_0 s_0 + \lambda_1 s_1 + \ldots + \lambda_{j-2} s_{j-2} + \lambda_{j-1} s_{j-1} &= 0,\\
\lambda_0 s_1 + \lambda_1 s_2 + \ldots + \lambda_{j-2} s_{j-1} + \lambda_{j-1} s_j &= 0,\\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot\\
\lambda_0 s_{j-2} + \lambda_1 s_{j-1} + \ldots + \lambda_{j-2} s_{2j-4} + \lambda_{j-1} s_{2j-3} &= 0,\\
\lambda_0 s_{j-1} + \lambda_1 s_j + \ldots + \lambda_{j-2} s_{2j-3} + \lambda_{j-1} s_{2j-2} &= r_{n-j}
\end{aligned}$$

From these equations we have, without difficulty,

$$r_{n-j} = \gamma_j \begin{vmatrix} s_0 & s_1 \ldots s_{j-1} \\ s_1 & s_2 \ldots s_j \\ \cdot & \cdot \quad \cdot \quad \cdot \\ s_{j-1} & s_j \ldots s_{2j-2} \end{vmatrix}, \quad A_j = \gamma_j \begin{vmatrix} s_0 & s_1 \ldots s_{j-2} & s_{j-1} \\ s_1 & s_2 \ldots s_{j-1} & s_j \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ s_{j-2} & s_{j-1} \ldots s_{2j-4} & s_{2j-3} \\ 1 & x \ldots x^{j-2} & x^{j-1} \end{vmatrix},$$

the value of γ_j being so far arbitrary. It appears, therefore, that the coefficient of the highest power of x in R_j differs by this multiplier only from the determinant $(s_0 s_2 s_4 \ldots s_{2j-2})$. We proceed to show that the sign of γ_j is positive. For this purpose we make use of the following relation connecting the successive values of the functions R and A :—

$$A_{k+1} R_k - R_{k+1} A_k = f(x). \qquad (2)$$

To prove this, substituting for $R_{k+1} R_k$, R_{k-1}, their values in terms of A and B in the relation $R_{k+1} = Q_k R_k - R_{k-1}$, we derive

$$A_{k+1} = Q_k A_k - A_{k-1}, \; B_{k+1} = Q_k B_k - B_{k-1};$$

by aid of which we readily obtain the following relations connecting the successive functions :—

$$A_{k+1} B_k - A_k B_{k+1} = A_k B_{k-1} - A_{k-1} B_k = \ldots = A_1 B_0 - A_0 B_1 = -1,$$

$$A_{k+1} R_k - A_k R_{k+1} = A_k R_{k-1} - A_{k-1} R_k = \ldots = A_1 R_0 - A_0 R_1 = f(x),$$

in which $R_1 = f'(x)$, $R_0 = f(x) = x^n + np_1 x^{n-1} + \ldots + p_n$.

Now, comparing the coefficients of the highest powers of x in (2): observing that x^n occurs only in $A_{k+1}R_k$, and making use of the determinant forms previously obtained, we have

$$\gamma_{k+1}\,(s_0 s_2 s_4 \,.\,.\,.\, s_{2k-2})\,\gamma_k\,(s_0 s_2 s_4 \,.\,.\,.\, s_{2k-2}) = 1,$$

or
$$\gamma_k \gamma_{k+1} = (s_0 s_2 s_4 \,.\,.\,.\, s_{2k-2})^{-2}.$$

Also, calculating the value of R_2 in the ordinary manner, we easily find

$$A_2 = \frac{1}{s_0^2}\begin{vmatrix} s_0 & s_1 \\ 1 & x \end{vmatrix};$$

whence it is seen that the value of γ_2 is $\dfrac{1}{s_0^2}$.

It follows, from the relation just established between any two successive values of γ, that $\gamma_3, \gamma_4, \ldots \gamma_j$, &c., are all positive squares, and therefore, finally, that r_{n-j}, the coefficient of the highest power of x in R_j, has the same sign as the determinant $(s_0 s_2 s_4 \,.\,.\,.\, s_{2j-2})$.

It should be noticed that there is only one way of obtaining a function of x, of the degree $n - j$, in the form $Af'(x) - Bf(x)$, where A and B are of the degrees $j - 1$ and $j - 2$, respectively, and $f(x)$ of the degree n; for this function being in general of the degree $n + j - 2$, in order that it may reduce to the degree $n - j$, the $2j - 2$ highest terms must vanish, and this is exactly the number of undetermined quantities in A and B at our disposal, since it is the ratios only of the coefficients we are concerned with. Sturm's remainders may therefore be obtained in this way with an undetermined multiplier.

The functions R_j, A_j, and B_j are functions of the differences of $x, \alpha_1, \alpha_2, \ldots \alpha_n$, and so, in other words, are semicovariants of $f(x)$, as may be seen by putting $x + \rho$ for x and $\alpha_r + \rho$ for α_r in the identity $R_j = A_j f'(x) - B_j f(x)$, noting that $f(x)$ and $f'(x)$ are unaltered, and hence that R_j, A_j, B_j must be independent of ρ as they are uniquely determined from $f(x)$, $f'(x)$ when j is assigned. Their actual expressions in terms of the differences of x and the roots can be readily inferred from the discussion in the following Article.

205. **Sylvester's Forms of Sturm's Functions.**—We make use of the notation employed in the preceding Article, and propose to show that the Sturmian remainder R_j differs only by the positive factor γ_j from the function V_j. We have

$$R_j \equiv A_j f'(x) - B_j f(x), \qquad (1)$$

where
$$R_j = r_0 + r_1 x + r_2 x^2 + \ldots + r_{n-j} x^{n-j},$$
$$A_j = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \ldots + \lambda_{j-1} x^{j-1},$$
$$B_j = \mu_0 + \mu_1 x + \mu_2 x^2 + \ldots + \mu_{j-2} x^{j-2};$$

also from the value of r_{n-j} above given we have immediately

$$r_{n-j} = \gamma_j \Sigma \nabla(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_j),$$

showing that the leading coefficients in R_j and V_j differ only by the factor γ_j. We now proceed to prove that the last coefficients in these functions differ only by the same factor. For this purpose, dividing the identity (1) by $f(x)$, substituting in it from the equation

$$\frac{f'(x)}{f(x)} = \frac{s_0}{x} + \frac{s_1}{x^2} + \frac{s_2}{x^3} + \ldots,$$

and comparing the coefficients, we find

$$\begin{aligned} \mu_0 &= \lambda_1 s_0 + \lambda_2 s_1 + \lambda_3 s_2 + \ldots + \lambda_{j-1} s_{j-2}, \\ \mu_1 &= \lambda_2 s_0 + \lambda_3 s_1 + \ldots + \lambda_{j-1} s_{j-3}. \\ &\ldots \\ \mu_{j-2} &= \lambda_{j-1} s_0. \end{aligned}$$

Also, putting $x = 0$ in (1) we have

$$r_0 = \lambda_0 p_{n-1} - \mu_0 p_n,$$

and substituting for μ_0 in terms of $\lambda_1, \lambda_2, \lambda_3$, &c.,

$$-\frac{r_0}{p_n} = \lambda_0 s_{-1} + \lambda_1 s_0 + \lambda_2 s_1 + \ldots + \lambda_{j-1} s_{j-2};$$

whence, giving to $\lambda_0, \lambda_1, \ldots \lambda_{j-1}$ the same values as in the calculation of r_{n-j}, we find

$$r_0 = (-1)^j p_n \gamma_j \begin{vmatrix} s_{-1} & s_0 & s_1 & \ldots & s_{j-2} \\ s_0 & s_1 & s_2 & \ldots & s_{j-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{j-2} & s_{j-1} & s_j & \ldots & s_{2j-3} \end{vmatrix}.$$

Now, referring to the calculation of Δ_j in Art. 203, and putting $\rho = 0$, or $\nu_r = \frac{1}{\alpha_r}$, in the value of Δ_j there found, we find for the determinant just written the value

$$\Sigma \frac{\nabla(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_j)}{\alpha_1 \alpha_2 \alpha_3 \ldots \alpha_j};$$

then, expressing the determinant as the product of two arrays, and giving p_n its value in terms of the roots, we have

$$r_0 = (-1)^{n-j} \gamma_j \Sigma \nabla(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_j) \alpha_{j+1} \alpha_{j+2} \ldots \alpha_n,$$

which was required to be proved.

Now R_j is, as we saw, a semicovariant, and

$$\therefore = \phi(\alpha_1 - x, \alpha_2 - x, \ldots \alpha_n - x);$$

therefore $r_0 = \phi(\alpha_1 \alpha_2 \ldots \alpha_n)$; so R_j is derived from r_0 by substituting $\alpha_r - x$ for α_r. Also, γ_j is a function of the differences of the roots.

$$\therefore R_j = (-1)^{n-j} \gamma_j \Sigma \nabla(\alpha_1, \alpha_2, \ldots \alpha_j)(\alpha_{j+1} - x)(\alpha_{j+2} - x) \ldots (\alpha_n - x)$$

$$\therefore R_j \equiv \gamma_j V_j.$$

EXAMPLES.

1. Using the notation of Arts. 204, 205, prove that the quotient of A_j by γ_j can be written as a symmetric function involving x and the roots: e.g.,

$$\frac{A_4}{\gamma_4} = \Sigma(\beta - \gamma)^2 (\gamma - \alpha)^2 (\alpha - \beta)^2 (x - \alpha)(x - \beta)(x - \gamma).$$

2. With the same notation prove that

$$B_j = \gamma_j \begin{vmatrix} s_0 & s_1 & s_2 & \dots & s_{j-1} \\ s_1 & s_2 & s_3 & \dots & s_j \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{j-2} & s_{j-1} & s_j & \dots & s_{2j-3} \\ 0 & T_1 & T_2 & \dots & T_{j-1} \end{vmatrix},$$

where $$T_j = s_0 x^{j-1} + s_1 x^{j-2} + s_2 x^{j-3} + \dots + s_{j-1}.$$

3. With the same notation, and denoting by U_n

$$\sum_{r=1}^{r=n} (\rho - \alpha_r)(x_1 + \alpha_r x_2 + \alpha_r^2 x_3 + \dots + \alpha_r^{n-1} x_n)^2,$$

prove that the discriminant of U_j may be determined by the equation $\gamma_j \Delta_j = A_j$, and show directly that if $A_j = 0$ for a certain value of x, A_{j-1} and A_{j+1} have opposite signs for the same value of x.

Section III.—Miscellaneous Theorems.

206. Reduction of the Quintic to the Sum of Three Fifth Powers.—This reduction can be effected by the solution of an equation of the third degree, as we proceed to show. Let

$$(a_0, a_1, a_2, a_3, a_4, a_5)(x, y)^5 \equiv b_1 (x - \beta_1 y)^5 + b_2 (x - \beta_2 y)^5 + b_3 (x - \beta_3 y)^5,$$

where $\beta_1, \beta_2, \beta_3$ are the roots of the equation

$$p_3 z^3 + p_2 z^2 + p_1 z + p_0 = 0.$$

Now, comparing coefficients in the two forms of the quintic,

$$a_0 = b_1 + b_2 + b_3, \quad -a_1 = b_1\beta_1 + b_2\beta_2 + b_3\beta_3,$$
$$a_2 = b_1\beta_1^2 + b_2\beta_2^2 + b_3\beta_3^2, \quad -a_3 = b_1\beta_1^3 + b_2\beta_2^3 + b_3\beta_3^3,$$
$$a_4 = b_1\beta_1^4 + b_2\beta_2^4 + b_3\beta_3^4, \quad -a_5 = b_1\beta_1^5 + b_2\beta_2^5 + b_3\beta_3^5;$$

whence

$$p_0 a_0 - p_1 a_1 + p_2 a_2 - p_3 a_3 = 0,$$
$$p_0 a_1 - p_1 a_2 + p_2 a_3 - p_3 a_4 = 0,$$
$$p_0 a_2 - p_1 a_3 + p_2 a_4 - p_3 a_5 = 0.$$

When these equations are taken in conjunction with the equation

$$p_0 + p_1 z + p_2 z^2 + p_3 z^3 = 0,$$

we have the following equation to determine $\beta_1, \beta_2, \beta_3$:—

$$C \equiv \begin{vmatrix} -1 & z & -z^2 & z^3 \\ a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \end{vmatrix} = 0.$$

When $\beta_1, \beta_2, \beta_3$ are determined by this equation, it will follow that any values of b_1, b_2, b_3 which satisfy three of the six equations above, satisfy the other three, and so b_1, b_2, b_3 may be found from the equations

$$\begin{aligned} b_1 \quad + b_2 \quad + b_3 \quad &= a_0, \\ b_1\beta_1 + b_2\beta_2 + b_3\beta_3 &= -a_1, \\ b_1\beta_1{}^2 + b_2\beta_2{}^2 + b_3\beta_3{}^2 &= a_2, \end{aligned}$$

and the solution thereby completed.

This important transformation of the quintic is a particular case of the following general theorem (proved in an exactly similar manner) due to Sylvester :—

Any homogeneous function of x, y, of the degree $2n-1$, can be reduced to the form

$$b_1(x-\beta_1 y)^{2n-1} + b_2(x-\beta_2 y)^{2n-1} + \ . \ . \ . + b_n(x-\beta_n y)^{2n-1}$$

by the solution of an equation of the n^{th} degree.

The cubic, C, in z, when written as a homogeneous equation in x, y (called the canonizant), equals

$$p_3(x-\beta_1 y)(x-\beta_2 y)(x-\beta_3 y),$$

and must be a covariant, because if the quintic is expressed in the form $u^5 + v^5 + w^5$, when transformed it is $u'^5 + v'^5 + w'^5$, where u', v', w' are the transformed values of u, v, w, but the transformed quintic is uniquely expressible in the form

$$u''^5 + v''^5 + w''^5,$$

and so u'', v'', w'', formed from the transformed equation, are equal to u', v', w', directly transformed from u, v, w, formed in a corresponding way for the original quintic, and therefore uvw is an absolute covariant. The canonizant is also easily seen to be the J invariant of the quadratic emanent, and so obtained by substituting in J,

$$\frac{\partial^4 U}{\partial x^4}, \frac{\partial^4 U}{\partial x^3 \partial y}, \ldots \frac{\partial^4 U}{\partial y^4}$$

for $a_0, a_1, \ldots a_4$, where U is the quintic, or otherwise it is seen to be the covariant whose source is obtained from J by altering $a_0, a_1, \ldots a_4$ to $a_5, a_4, \ldots a_1$.

When the cubic C has a root equal to infinity, using the forms $y - \beta_1 x$, $y - \beta_2 x$, $y - \beta_3 x$, we get an equation for $\beta_1, \beta_2, \beta_3$ of which one root is zero.

When the cubic C has two equal roots, or $\beta_1 = \beta_2$, the reduction to three fifth powers is not possible, as we cannot satisfy any three of the equations for b_1, b_2, b_3, for they involve b_1, b_2 in the form $b_1 + b_2$ only. Putting $\beta_2 = \beta_1 + \epsilon$, $\beta_3 = \beta_1 + \eta$, and solving for b_1, b_2, b_3 in terms of β, ϵ, η, and writing $u, u - \epsilon y, u - \eta y$, for $x - \beta_1 y$, $x - \beta_2 y$, $x - \beta_3 y$, we get in the limit when $\epsilon = 0$, that the quintic may be expressed in the form $Au^5 + Bu^4 v + Cv^5$. Further, when the canonizant has all its roots equal, we find that the quintic may be expressed in the form $Au^5 + Bu^3 v^2$, by getting the limit of the last form when $\eta = 0$.

If the canonizant vanishes identically, we can find q_0, q_1, q_2, so that

$$q_0 a_0 - q_1 a_1 + q_2 a_2 = 0,\ q_0 a_1 - q_1 a_2 + q_2 a_3 = 0,$$
$$q_0 a_2 - q_1 a_3 + q_2 a_4 = 0,\ q_0 a_3 - q_1 a_4 + q_2 a_5 = 0;$$

and if we take β_1, β_2 equal to the roots of $q_0 + q_1 z + q_2 z^2 = 0$, we can express the quintic in the form $b_1 (x - \beta_1 y)^5 + b_2 (x - \beta_2 y)^5$, or as the sum of two fifth powers. Following the same method as for the quintic, if we try to express the quartic as the sum of two fourth powers, we get $J = 0$; and if we try to express the sextic as the sum of three sixth powers, we get a determinant $= 0$

whose rows are (a_0, a_1, a_2, a_3), (a_1, a_2, a_3, a_4), (a_2, a_3, a_4, a_5), (a_3, a_4, a_5, a_6), and so on for a quantic of degree $2n$, we express as a determinant the condition that it may be expressible as the sum of $2\ n^{th}$ powers of n linear functions. All such conditions are invariants for the corresponding quantics, as may also be verified by performing the general transformation $x = lx' + my'$, $y = l'x' + m'y'$, by the successive transformations $x = \alpha x_1$, $y = \beta y_1$; $x_1 = x_2 + \gamma y_2$, $y_1 = y_2$; $x_2 = x'$, $y_2 = y' + \delta x'$, and noting that if I is one of the above conditions,

$$I' = I_2 = I_1 = (\alpha\beta)^\kappa I.$$

207. **Quartics and Cubics Transformable into each other.**—If two quantics U, U' can be transformed into each other, it is obviously necessary that corresponding invariants I, I' should be connected by the relation $I' = M^\kappa I$, where κ is their weight and M the same constant for all such pairs of invariants, and also that if any covariant vanishes identically for U, the corresponding one for U' also vanishes identically. These conditions are sufficient for cubics and quartics.

Two Cubics.—(*a*) If Δ does not vanish, and so the cubic U has not a square factor, then by the method of the preceding section, or of Vol. I., page 111, assuming that a_0 is not zero, U may be expressed as the sum of the cubes of two linear functions u, v. As Δ does not vanish by hypothesis, neither does Δ', so U' similarly $= u'^3 + v'^3$. Hence U may be transformed to U by $u = \omega u'$, $v = \theta v'$, or $u = \omega v'$, $v = \theta u'$, where $\omega^3 = \theta^3 = 1$.

Note if $a_0 = 0$ for a quantic $U(x, y)$ we can transform U to a form in which a_0 does not vanish by putting $x = X$, $y = lX + Y$, where l is chosen so that $U(1, l)$ does not vanish. Thus a cubic U for which a_0 vanishes may be transformed to one for which a_0 does not vanish, then by the method of Vol. I., page 111, expressed as the sum of two cubes, and by putting $X = x$, $Y = y - lx$, the original cubic is expressed as the sum of two cubes.

(*b*) If H_x vanishes identically, $U \equiv u^3$, and as by hypothesis H'_x also vanishes identically, and $\therefore U' = u'^3$, U may be transformed to U' by taking $u = \omega u'$, where $\omega^3 = 1$, with any other linear relation between x, y and x', y'.

(*c*) If $\Delta = 0$ and $H_x \not\equiv 0$, U is of form $u^2 v$, and as $\Delta' = 0$, $H_x' \not\equiv 0$, U' also $= u'^2 v'$, $\therefore U$ may be transformed to U' by $u = \omega u'$, $v = \theta v'$, where $\omega^2 \theta = 1$.

The three classes of cubics, such that all cubics of the same class are transformable into each other by a linear transformation, are distinguished by (*a*) $\Delta \neq 0$, (*b*) $H_x \equiv 0$, (*c*) $\Delta = 0$, $H_x \not\equiv 0$.

Two Quartics.—(*a*) If $\Delta = I^3 - 27J^2 \neq 0$, so that the quartic U has not a square factor, using the method and notation of section 183, and assuming a_0 does not vanish,

$$4\frac{U}{a_0} = \frac{u^2 - v^2}{\lambda - \mu} = \frac{1}{\lambda - \mu}\left\{\frac{4u_1^2 u_2^2}{\mu - \nu} - \frac{(u_1^2 + u_2^2)^2}{\lambda - \nu}\right\}$$

$$= \frac{1}{(\nu - \lambda)(\lambda - \mu)}\left\{u_1^4 + u_2^4 - \frac{2(2\lambda - \mu - \nu)}{\mu - \nu} u_1^2 u_2^2\right\}$$

$$\therefore U = \frac{a_0^3}{64(\rho_3 - \rho_1)(\rho_1 - \rho_2)}\left\{u_1^4 + u_2^4 + \frac{6\rho_1}{\rho_2 - \rho_3} u_1^2 u_2^2\right\}.$$

Now as $\Delta' \neq 0$, U' may be similarly expressed, and as by hypothesis $I' = M^4 I$, $J' = M^6 J$, $\therefore$ the roots of $4\rho'^3 - I'\rho' + J' = 0$ are equal to the corresponding roots of $4\rho^3 - I\rho + J = 0$ multiplied by M^2, and $\therefore \rho_1' / (\rho'_2 - \rho'_3) = \rho_1 / (\rho_2 - \rho_3)$.

Hence U may be transformed to U' by taking $u_1 = \omega u_1'$, $u_2 = \theta u_2'$, or $u_1 = \omega u_2'$, $u_2 = \theta u_1'$, where $\omega^4 = \theta^4 = a_0'^3 / M^4 a_0^3$.

When either a_0 or a'_0 vanish we can proceed by first transforming U or U' as above to a form for which a_0 or a_0' does not vanish.

Reverting now to the notation in which we regard u, v as linear functions of x, y, we see that all quartics for which $I^3 - 27J^2 \neq 0$ can be expressed in the form $u^4 + v^4 + 6\lambda u^2 v^2$, where λ is the same for all quartics which have the same absolute invariant I^3 / J^2, an absolute invariant being one which is unaltered by linear transformation. For all such quartics λ is the same, because if $I^3 / J^2 = I'^3 / J'^2$, taking $I' = M^4 I$, $\therefore J'^2 = M^{12} J^2$, $\therefore J' = \pm M^6 J$, and if the negative sign occurs, put $-M^2$ for M^2, $\therefore I' = M^4 I$, $J' = M^6 J$.

(*b*) if $H_x \equiv 0$, $U = u^4$, and as by hypothesis $H'_x \equiv 0$, $\therefore$ $U' = u'^4$, and so U may be transformed to U' by taking $u = \omega u'$, where $\omega^4 = 1$, along with any other arbitrary linear equation between x, y, x', y'.

(*c*) If $\Delta = 0$, $H_x \not\equiv 0$, U must be of form $u^2v(4bu + 6cv)$. For such form writing x for u and y for v, $I = 3c^2$, $J = -c^3$, $G_x = 2b^2x^5(bx + 3cy)$.

If $\therefore$ $\Delta = 0$, $H_x \not\equiv 0$, $I \neq 0$, $J \neq 0$, $G_x \not\equiv 0$, U is of form $u^2v(4bu + 6cv)$, where $b \neq 0$, $c \neq 0$, and U' is of same form, and $\therefore$ U may be transformed to U' by taking $u = lu'$, $v = mv'$, where $l^3m = b' / b$, $l^2m^2 = c' / c$.

(*d*) If $I = 0$, $J = 0$, $H_x \not\equiv 0$, $G_x \not\equiv 0$, then $c = 0$, $b \neq 0$. $\therefore$ $U = 4bu^3v$, $U' = 4b'u'^3v'$, and U may be transformed to U' by taking $u = lu'$, $v = mv'$, with $l^3m = b' / b$.

(*e*) If $\Delta = 0$, $I \neq 0$, $J \neq 0$, $H_x \not\equiv 0$, $G_x \equiv 0$, then $c \neq 0$, $b = 0$. $\therefore$ $U = 6cu^2v^2$, $U' = 6c'u'^2v'^2$, and U may be transformed to U' by taking $u = lu'$, $v = mv'$ or $u = lv'$, $v = mu'$, with $l^2m^2 = c'/c$.

Thus quartics may be divided into five classes, and quartics of the same class may be transformed into each other by a linear transformation, if corresponding invariants and covariants are connected by the relations given at the beginning of this article.

208. **Number of Absolute Invariants of any Quantic.**—We proceed now to examine how the number of absolute invariants of any binary quantic is connected with the number of ordinary invariants, and how far a limit can be determined to either of these numbers. Transforming the quantic

$$(a_0, a_1, a_2, \ldots a_n)(x, y)^n$$

by the substitution

$$x = \lambda X + \mu Y, \quad y = \lambda' X + \mu' Y;$$

if the new form be

$$(A_0, A_1, A_2, \ldots A_n)(X, Y)^n,$$

we have by the comparison of coefficients $n + 1$ equations expressing $A_0, A_1, \ldots A_n$ as follows :—

$$A_0 = (a_0, a_1, a_2, \ldots a_n)(\lambda, \lambda')^n, \ \ldots \ A_j = \frac{\Gamma_{(j)}}{\Gamma_{(n)}}\Delta^{n-j}A_n, \ \ldots,$$

$$A_n = (a_0, a_1, a_2, \ldots a_n)(\mu, \mu')^n,$$

where $\Delta = \lambda\frac{\partial}{\partial\mu} + \lambda'\frac{\partial}{\partial\mu'}$, $\Gamma_{(j)} = 1 \cdot 2 \cdot 3 \ \ldots \ j$, $\Gamma_{(o)} = 1$.

Now, eliminating $\lambda, \mu, \lambda', \mu'$, we obtain, among the new and old coefficients, $n-3$ independent relations, which number is therefore a superior limit to the number of absolute invariants. But if $(\lambda\mu' - \lambda'\mu)$ be admitted when $\lambda, \mu, \lambda', \mu'$ are excluded by elimination, we must add the equation $\lambda\mu' - \lambda'\mu = M$ to the $n+1$ equations given above; and when the elimination is now completed, we have $n-2$ independent relations. We will assume, as suggested by our previous investigations, that these relations can be reduced to the form

$$\phi_r(A_0, A_1, A_2, \ldots A_n) = M^j\phi_r(a_0, a_1, a_2, \ldots a_n); \ \text{(Art. 171)}$$

and we have therefore $n-2$ independent ordinary invariants $\phi_1, \phi_2, \phi_3, \ldots \phi_{n-2}$. Eliminating M we obtain, as above stated, $n-3$ relations connecting the two sets of coefficients, and this, therefore, is the number of *independent* absolute invariants. It is not true, in general, that every invariant can be expressed as a *rational* function of the invariants $\phi_1, \phi_2, \phi_3, \ldots \phi_{n-2}$; and consequently we have not obtained a superior limit to the number of ordinary invariants by this investigation (see Note E).

209. **Number of Seminvariants of a Quantic.**—Every seminvariant can be expressed rationally in terms of a_0 and $n-1$ functions of the coefficients which are either invariants or seminvariants. For, removing the second term from the equation

$$U_n \equiv (a_0, a_1, a_2, \ldots a_n)(x, 1)^n = 0,$$

the new coefficients are easily obtained by substituting for h its value $-\frac{a_1}{a_0}$ (Art. 35). As these coefficients, when divided by a_0, are symmetric functions of the differences of the roots, they must be invariants or seminvariants when multiplied by a power of a_0; also, every other symmetric function of the differences of the roots must be a rational function of the same quantities, but not necessarily *integral* when multiplied by a_0^{ϖ}; consequently, we have not obtained a superior limit to the number of independent seminvariants (or, which is the same thing, covariants) by this investigation. It has been proved, however, by Gordan that the number of seminvariants of any quantic is finite.

As an illustration of the preceding, we give the values of A_2, A_3, A_4, A_5, A_6 in a reduced form—

$$a_0A_2 = H, \quad a_0^2A_3 = G, \quad a_0^3A_4 = a_0^2I - 3H^2, \qquad \text{(Art. 37)}$$

$$a_0^4A_5 = a_0^2F - 2GH,$$

$$a_0^5A_6 = 45H^3 - 15a_0^2HI + 10G^2 + a_0^4I_2,$$

where $$F = a_0^2a_5 - 5a_0a_1a_4 + 2a_0a_2a_3 - 6a_1a_2^2 + 8a_1^2a_3,$$

$$I_2 = a_0a_6 - 6a_1a_5 + 15a_2a_4 - 10a_3^2,$$

F being a seminvariant, and I_2 an invariant of the sextic U_6. (Exs. 4, 6, p. 104.) We have, therefore, proved that every seminvariant of the sextic can be expressed in the form

$$a_0^{-r}\Psi(a_0, F, G, H, I, I_2),$$

where Ψ is a rational and integral function; and, consequently, every covariant when multiplied by a power of U_6 may be expressed as follows:—

$$\Psi(U_6, F_x, G_x, H_x, I_x, I_2).$$

We conclude with the following important observation:— *When a rational and integral function of several seminvariants is formed so that the result is divisible by a_0, a new seminvariant is obtained which is considered distinct from the others.*

210. **Hermite's Law of Reciprocity.**—THEOREM. *A quantic $(a_0, a_1, \ldots a_n)(x, y)^n$, of degree n, has as many covariants of the order ϖ in the coefficients as a quantic $(a_0, a_1, \ldots a_{\varpi})(x, y)^{\varpi}$, of degree ϖ, has covariants of the order n in the coefficients.*

This theorem can be shown to depend on Cayley's theorem (Art. 165) as to the number of distinct seminvariants of given order and weight of any quantic. When for a quantic of the n^{th} degree an integral homogeneous function of the coefficients is formed containing all possible terms of order ϖ and weight κ which can be made out of the coefficients $a_0, a_1, a_2, \ldots a_n$, it can be proved that there will be exactly the same number of terms in the corresponding expression of order n and same weight κ, which can be formed for a quantic of degree ϖ from the coefficients $a_0, a_1, a_2, \ldots a_\varpi$. For this purpose Mr. Ferrers has employed a mechanical method of transformation term by term, which will be readily understood from a particular application: Let us suppose that an expression of order 8 and weight 22 of a quintic contains the term $a_1{}^2 a_2 a_3{}^3 a_4 a_5$ (which we write $a_1 a_1 a_2 a_3 a_3 a_3 a_4 a_5$); and let the weights of the successive factors be represented by points arranged horizontally as follows:—

If now the points be counted in vertical in place of horizontal order, we obtain the term $a_8 a_6 a_5 a_2 a_1$, of order 5 and weight 22. It is clear that two terms thus derived from one another have always equal weights, since the total number of points counted in both cases is the same. We see therefore that to any term of order 8 and weight 22 derived from the coefficients of a quintic corresponds a term of order 5 and weight 22 similarly derived from the coefficients of an octavic; this relation is reciprocal, so that for each term of either function there exists a corresponding term of the other, and if one list of terms be complete, the derived list must also be complete. In applying this transformation it must be observed that if the term to be transformed does not contain the coefficient with highest suffix of the corresponding quantic, the order of the derived term will be deficient, and the factor a_0 with proper index must be supplied, this of course not affecting the weight. Since therefore the two corresponding expressions thus derivable from one another contain the same number of terms, we may represent this result by the notation

$$N(\varpi, \kappa, n) = N(n, \kappa, \varpi).$$

The same is true for similar functions whose weight is one less in each case. We have therefore

$$N(\varpi, \kappa, n) - N(\varpi, \kappa - 1, n) = N(n, \kappa, \varpi) - N(n, \kappa - 1, \varpi),$$

from which, by Cayley's theorem (p. 105), it follows that the number of seminvariants of order ϖ and weight κ which can be made out of $a_0, a_1, a_2, \ldots a_n$ is equal to the number of order n and weight κ which can be made out of $a_0, a_1, a_2, \ldots a_\varpi$.

Hermite's theorem as to covariants follows immediately, since the corresponding seminvariants can be taken as leading coefficients of covariants; and, moreover, since $n\varpi - 2\kappa = \varpi n - 2\kappa$, *the degrees of two corresponding covariants are equal.* As a particular case, also, we see that *to an invariant of one quantic corresponds an invariant of the other.*

EXAMPLES.

1. Show that the terms written with literal coefficients only which occur in the resultant of a cubic, by the transformation above described, supply the literal terms of the cubic invariant of the quartic.

2. From the seminvariant of a quintic in Ex. 4, p. 104, derive the literal terms of the corresponding seminvariant of a cubic; and show that to the quintic-covariant of the former corresponds the product of H_x and G_x of the cubic.

3.—Show that quantics of the degree $2m$ alone have invariants of the second order in the coefficients.

For the only invariants of a quadratic are of the type Δ_m, whose order in the coefficients is $2m$, Δ being the discriminant.

211. **Reciprocal and Orthogonal Linear Transformation—Contravariants.**—When the coordinates of a point are transformed by a linear transformation, the tangential

coordinates of a line and the operating symbols $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are both transformed by the same new linear transformation, which is said to be reciprocal to the first.

Let the linear transformation be

$$\left.\begin{aligned} x &= a_1X + b_1Y + c_1Z, \\ y &= a_2X + b_2Y + c_2Z, \\ z &= a_3X + b_3Y + c_3Z \end{aligned}\right\} ; \qquad (1)$$

whence any line $\lambda x + \mu y + \nu z$ becomes, by transformation, $LX + MY + NZ$, where

$$\left.\begin{aligned} L &= a_1\lambda + a_2\mu + a_3\nu, \\ M &= b_1\lambda + b_2\mu + b_3\nu, \\ N &= c_1\lambda + c_2\mu + c_3\nu \end{aligned}\right\} ; \qquad (2)$$

also
$$\frac{\partial}{\partial X} = \frac{\partial x}{\partial X}\frac{\partial}{\partial x} + \frac{\partial y}{\partial X}\frac{\partial}{\partial y} + \frac{\partial z}{\partial X}\frac{\partial}{\partial z},$$

or, substituting for $\frac{\partial x}{\partial X}, \frac{\partial y}{\partial X}, \frac{\partial z}{\partial X}$ their values,

$$\frac{\partial}{\partial X} = a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z},$$

and similarly

$$\frac{\partial}{\partial Y} = b_1\frac{\partial}{\partial x} + b_2\frac{\partial}{\partial y} + b_3\frac{\partial}{\partial z}, \quad \frac{\partial}{\partial Z} = c_1\frac{\partial}{\partial x} + c_2\frac{\partial}{\partial y} + c_3\frac{\partial}{\partial z};$$

whence L, M, N and the symbols $\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}$ follow the same laws of transformation, and consequently λ, μ, ν and $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ also; in fact, from equations (2) this transformation is

$$\begin{aligned} \Delta\lambda &= A_1L + B_1M + C_1N, \\ \Delta\mu &= A_2L + B_2M + C_2N, \\ \Delta\nu &= A_3L + B_3M + C_3N, \end{aligned}$$

where $\Delta = (a_1b_2c_3), \quad A_1 = \frac{\partial\Delta}{\partial a_1}, \quad B_1 = \frac{\partial\Delta}{\partial b_1}$, &c., &c.

This linear transformation is said to be reciprocal to the transformation (1) whose modulus is Δ, its coefficients being

$$\frac{1}{\Delta}\frac{\partial \Delta}{\partial a_1}, \quad \frac{1}{\Delta}\frac{\partial \Delta}{\partial b_1}, \quad \frac{1}{\Delta}\frac{\partial \Delta}{\partial c_1}, \text{ \&c.}$$

The variables x, y, z, and $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ are said to be *contragredient* to each other, for a linear transformation of x, y, z leads to a linear transformation of the symbols $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$, which, although not the same, is connected with the first in the manner already explained.

We next define "orthogonal" transformation. If, in the equations (1) above written, we have among the coefficients the relations

$$a_1^2 + a_2^2 + a_3^2 = 1, \quad b_1^2 + b_2^2 + b_3^2 = 1, \quad c_1^2 + c_2^2 + c_3^2 = 1,$$
$$a_1b_1 + a_2b_2 + a_3b_3 = 0, \quad a_1c_1 + a_2c_2 + a_3c_3 = 0, \quad b_1c_1 + b_2c_2 + b_3c_3 = 0,$$

the transformation is said to be *orthogonal*. These conditions are fulfilled, for example, by the direction-cosines which enter into the relations between the coordinates of a point referred to two different sets of rectangular axes in solid geometry. In such a transformation it is clear that we have the relation

$$x^2 + y^2 + z^2 = X^2 + Y^2 + Z^2,$$

and that the new variables are expressed as follows in terms of the old :—

$$X = a_1x + a_2y + a_3z, \quad Y = b_1x + b_2y + b_3z, \quad Z = c_1x + c_2y + c_3z.$$

Also, if the modulus of transformation written as a determinant be squared, each of the elements contained in the principal diagonal is equal to unity, and all the other elements vanish.

When a quantic in x, y, z is transformed, any function involving the coefficients of the original quantic, together with other variables which are transformed by the reciprocal substitution above explained, is said to be a *contravariant* if it differs only by a power of the modulus of transformation from the corresponding function of the transformed coefficients and variables. The condition, for example, that a line $\lambda x + \mu y + \nu z$ should touch a conic given by an equation in trilinear coordinates is a contravariant. The theory of contravariants can be included under that of invariants by considering the combined system composed of the given quantic and $\lambda x + \mu y + \nu z$. For instance, in Prop. III. of Art. 175, if we substitute $x\lambda + y\mu + z\nu$ for $xy' - x'y$ and U is a quantic in x, y, z, we derive by the method given there contravariants which are called *evectants*.

It may be observed that in the case of binary quantics, contravariants and covariants are not essentially distinct, as a contravariant may be changed to a covariant by substituting x, $-y$ for μ, λ respectively, or vice versa, in a way we have frequently used, by substituting $\dfrac{\partial}{\partial y}$ and $-\dfrac{\partial}{\partial x}$ for x, y respectively in a covariant.

The treatment of questions in n variables is facilitated by the conventional rule that when a suffix occurs twice in a product, the product is to be summed for all values of the suffix from 1 to n. Thus a linear transformation from $x_1, x_2, \ldots x_n$ to $x_1', x_2', \ldots x_n'$ is expressed by $x_\alpha = l_{\alpha\beta}x_\beta'$, where as β occurs twice $x_\alpha = l_{\alpha\beta}x_\beta'$ stands for $x_\alpha = l_{\alpha 1}x_1' + l_{\alpha 2}x_2' + \ldots + l_{\alpha n}x'_n$. In the determinant $(l_{\alpha\beta})$, called the modulus of transformation M, if $L_{\beta\alpha}$ equals the minor corresponding to $l_{\alpha\beta}$ divided by M, the inverse substitution is expressed by $x_\alpha' = L_{\alpha\beta}x_\beta$. The condition that the transformation should be orthogonal is $l_{\alpha\beta}l_{\alpha\gamma} = 0$, unless $\beta = \gamma$, when it equals 1. Thus in an orthogonal transformation $x_\alpha x_\alpha = l_{\alpha\beta}x_\beta' l_{\alpha\gamma}x_\gamma'$, where α, β, γ are each summed for all values from 1 to n, keeping β, γ fixed and summing α, $l_{\alpha\beta}l_{\alpha\gamma} = 0$, unless $\beta = \gamma$ and then it equals 1, $\therefore x_\alpha x_\alpha = x_\beta' x_\beta'$. Moreover, in an orthogonal transformation, multiply $x_\alpha = l_{\alpha\beta}x_\beta'$ by $l_{\alpha\gamma}$, and summing, $l_{\alpha\gamma}x_\alpha = l_{\alpha\beta}l_{\alpha\gamma}x_\beta' = x_\gamma'$, $\therefore l_{\alpha\gamma} = L_{\gamma\alpha}$.

In the general transformation, if ξ_α represent a tangential variable, so that $\xi_\alpha x_\alpha = \xi_\alpha' x_\alpha'$, $\therefore \xi_\alpha x_\alpha = \xi_\alpha l_{\alpha\beta}x_\beta' = \xi_\beta' x_\beta'$, so $\xi_\beta' = l_{\alpha\beta}\xi_\alpha$ gives the reciprocal substitution. Also $\dfrac{\partial}{\partial x_\alpha'} = l_{\beta\alpha}\dfrac{\partial}{\partial x_\beta}$, so $\dfrac{\partial}{\partial x_\alpha}$ is cogredient with ξ_α. In an orthogonal transformation $\xi_1, \xi_2, \ldots \xi_n$ are cogredient with $x_1, x_2, \ldots x_n$.

Again $a_{\alpha\beta}x_\alpha x_\beta$, taking $a_{\alpha\beta} = a_{\beta\alpha}$, represents a homogeneous function of the second degree, and its discriminant Δ is the determinant $(a_{\alpha\beta})$. If we transform by the general linear transformation above, $a_{\alpha\beta}x_\alpha x_\beta = a_{\alpha\beta}l_{\alpha\gamma}l_{\beta\delta}x_\gamma' x_\delta'$, $\therefore a'_{\gamma\delta} = a_{\alpha\beta}l_{\alpha\gamma}l_{\beta\delta}$. Now the product of two determinants $(b_{\alpha\beta})$, $(c_{\alpha\beta})$ equals $(b_{\alpha\gamma}c_{\beta\gamma})$ or $(b_{\gamma\alpha}c_{\gamma\beta})$ or $(b_{\alpha\gamma}c_{\gamma\beta})$ according as we multiply rows by rows, or columns by columns or rows in $(b_{\alpha\beta})$ by rows in $(c_{\alpha\beta})$ after altering rows to columns. Hence $\Delta' =$

$(a'_{\gamma\delta}) = (a_{\alpha\beta} l_{\alpha\gamma} \,.\, l_{\beta\delta}) = (a_{\alpha\beta} l_{\alpha\gamma}) \,.\, (l_{\beta\delta})$, where β is not summed, $= (a_{\alpha\beta})(l_{\alpha\gamma})(l_{\beta\delta})$, where α, β are not summed, $= \Delta M^2$.

Similarly a homogeneous function of the third degree in n variables is expressed by $a_{\alpha\beta\gamma} x_\alpha x_\beta x_\gamma$, all the values of $a_{\alpha\beta\gamma}$ obtained by permuting $\alpha\beta\gamma$ being assumed equal. Similarly one of the fourth degree $U_4 = a_{\alpha\beta\gamma\delta} x_\alpha x_\beta x_\gamma x_\delta$, where all the values of $a_{\alpha\beta\gamma\delta}$, obtained by permuting $\alpha\beta\gamma\delta$, are assumed equal. To find $\partial U_4 / \partial x_\alpha$, we must observe that as α, β, γ, δ are to be summed for all values from 1 to n, the suffix α will occur in the place occupied by every suffix in $a_{\alpha\beta\gamma\delta}$. Thus $\partial U_4/\partial x_\alpha = a_{\alpha\beta\gamma\delta} x_\beta x_\gamma x_\delta + a_{\beta\alpha\gamma\delta} x_\beta x_\gamma x_\delta + a_{\beta\gamma\alpha\delta} x_\beta x_\gamma x_\delta + a_{\beta\gamma\delta\alpha} x_\beta x_\gamma x_\delta$ $= 4 a_{\alpha\beta\gamma\delta} x_\beta x_\gamma x_\delta$. Similarly

$$\frac{\partial^2 U_4}{\partial x_\alpha \partial x_\beta} = 4 \,.\, 3 \; a_{\alpha\beta\gamma\delta} x_\gamma x_\delta, \quad \frac{\partial^3 U}{\partial x_\alpha \partial x_\beta \partial x_\gamma} = 4 \,.\, 3 \,.\, 2 \,.\, a_{\alpha\beta\gamma\delta} x_\delta,$$

and $\dfrac{\partial^4 U}{\partial x_\alpha \partial x_\beta \partial x_\gamma \partial x_\delta} = 4\,!\; a_{\alpha\beta\gamma\delta}$, and so on for homogeneous functions of n variables of higher degrees.

Miscellaneous Examples.

1. Every quantic of an odd degree has a quadratic covariant of the second order in the coefficients.

For every quantic of an even degree has an invariant of the second order in the coefficients (Art. 177), which may be written in the form $U_D(U)$ or $(1, 2)^n U_1 U_2$, and this invariant of the quantic whose degree is $2m$ will be a seminvariant of one whose degree is $2m + 1 \equiv n$. The covariant therefore which has this seminvariant as leader will be a quadratic, since $n\varpi - 2\kappa = 2$, κ being $= n - 1$ and $\varpi = 2$.

2. Every quantic of an odd degree $2m + 1 \equiv n$ has a linear covariant of the degree n in the coefficients when n is greater than 3.

For, if $I\,(x, y)^2$, be the quadratic covariant of the preceding example, we have

$$I_D^m(U) \equiv L_0 x + L_1 y,$$

a linear covariant, the order of L_0 and L_1 being n. It is here assumed that L_0 and L_1 are not identically zero, as they are for the cubic.

3. Every quantic of an odd degree has an invariant of the fourth order in the coefficients of the form $A a_n{}^2 + 2 B a_n + C$.

The discriminant $I\,(x, y)^2$ is the required invariant.

4. Every quantic of odd degree n has a seminvariant of the third order in the coefficients which is the leader of a covariant of the n^{th} degree.

For, differentiating with regard to a_n the discriminant obtained in the preceding example, we have, for the resulting seminvariant, $\varpi = 3$, $\kappa = n$, and consequently $\rho = n\varpi - 2\kappa = n$, which is therefore the degree of the covariant of which $\dfrac{\partial \Delta}{\partial a_n}$ is the leader.

The series of seminvariants obtained in this way for the odd quantics is important, the order in the coefficients being low.

5. Quantics of the degree $4m$ have invariants of the third order in the coefficients.

For cubics have invariants of the type Δ^m, of the order $4m$ in the coefficients, Δ being the discriminant. This and the next four examples are immediate deductions from Hermite's Law of Reciprocity (Art. 210).

6. Quantics of the degree m have as many invariants of the fourth order as there are solutions in positive integers of the equation $2p + 3q = m$. A quintic, for example, has one, a sextic two, a septimic one, an octavic two; and so on.

For quartics have invariants of the type $I^p J^q$, which is of the order $2p + 3q = m$ in the coefficients.

7. Every quantic of the degree $2p + q$ has a covariant of the second order in the coefficients. In particular, when $q = 1$, every quantic of odd degree has a quadratic covariant of the second order in the coefficients (cf. Ex. 1).

For quadratics have covariants of the type $\Delta^p U_2{}^q$, which is of the order $2p + q$ in the coefficients.

8. Every binary quantic of an odd degree greater than 3 has a linear covariant of the fifth order in the coefficients.

For a quintic has an invariant I_4 of the fourth order, the discriminant of I_x, also covariants of the fifth and seventh orders, viz. L_x (Ex. 2) and $M_x \equiv L_D I_x$; from these we form the covariants $I_4{}^{p-1}L_x$, of order $4p + 1$, and $I_4{}^{p-2}M_x$, of order $4p - 1$; but every prime number is of the form $4p \pm 1$. —HERMITE.

9. Every quantic of the degree $4p + 2$ has a quadratic covariant of the third order in the coefficients.

For a cubic has a quadratic covariant of the type $\Delta^p H_x$, of the order $4p + 2$ in the coefficients.

10. When the quintic $(a_0, a_1, a_2, a_3, a_4, a_5)(x, y)^5$ has a triple factor, prove that the covariant I_x is a perfect square, and the covariant J_x a perfect cube, the linear factor being the triple factor of the quintic in both cases.

11. When the quintic has two double factors, the remaining factor is a single factor of J_x.

12. If $U_x \equiv (a_0, a_1, a_2, \ldots a_n)(x, y)^3$ prove that the resultant of U_x and the covariant G_x is the discriminant of U cubed; that is, $R(U_x, G_x) = 2\Delta^3(U_x)$; and prove also $R(U_x, H_x) = k\Delta^2(U_x)$.

Express H_x and G_x in terms of the semicovariants $U_1, \ldots U_{n-1}, U_0, U$.

13. Express the combinant P^2 of two cubics in terms of the other invariants, p. 162. *Ans.* $P^2 = 16I_{13} - 4I_{22} = \Phi + 24I_{13}$,

where $I_{11}, I_{13}, \ldots$ &c., are the invariants of the three Hessians of Art. 192.

14. When the quintic has a triple root, the following symmetric functions of the roots vanish:—

$$\Sigma(\alpha_1 - \alpha_2)^2 \nabla(\alpha_3, \alpha_4, \alpha_5), \quad \Sigma(\alpha_1 - \alpha_2)^4 \nabla(\alpha_3, \alpha_4, \alpha_5).$$

15. Two quadratics, U, V, in x, y may be expressed in one of the forms

(i) $Au^2 + Bv^2$, $A'u^2 + B'v_2$. (ii) uv, uw. (iii) $U \equiv kV$,

where u, v, w are linear functions of x, y.

Putting
$$U \equiv a_0x^2 + 2b_1xy + a_2y^2 \equiv A(x - \alpha y)^2 + B(x - \beta y)^2,$$
$$V \equiv b_0x^2 + 2b_1xy + b_2y^2 \equiv A'(x - \alpha y)^2 + B'(x - \beta y)^2.$$

$$\begin{aligned} a_0 &= A + B & b_0 &= A' + B'. \\ -a_1 &= A\alpha + B\beta. & -b_1 &= A'\alpha + B'\beta \\ a_2 &= A\alpha^2 + B\beta^2 & b_2 &= A'\alpha^2 + B'\beta^2. \end{aligned} \qquad (1)$$

If α, β are roots of $px^2 + qxy + ry^2 = 0$, $pa_2 - qa_1 + ra_0 = 0$, $pb_2 - qb_1 + rb_0 = 0$, and $\therefore$ α, β are roots of

$$K \equiv \begin{vmatrix} x^2 & -xy & y^2 \\ a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 \end{vmatrix} = 0.$$

If the roots of $K = 0$ are different, we get A, B, A', B' from the first two equations of each set (1) and establish the result (i). Putting $\beta = \alpha + \epsilon$ in the values of A, B, A', B' and finding the limit when $\epsilon = 0$ of the forms assumed for U, V, we get (ii). When $K \equiv 0$, $a_0 = kb_0$, $a_1 = kb_1$, $a_2 = kb_2$ and $U \equiv kV$. We see that $K \equiv J(V, U)$, and its factors are u, v.

16. If the coefficients of three quadratics

$$a_0x^2 + 2a_1xy + a_2y^2,\ b_0x^2 + 2b_1xy + b_2y^2,\ c_0x^2 + 2c_1xy + c_2y^2$$

be connected by the relation

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{vmatrix} = 0,$$

prove that they may be in general reduced by linear transformations to the forms

$$A_1X^2 + B_1Y^2, \quad A_2X^2 + B_2Y^2, \quad A_3X^2 + B_3Y^2.$$

The determinant here written is the condition that the three quadratics should determine a system of points or lines in involution.

17. If U, V are two homogeneous functions of the second degree in n variable with real coefficients, and if V is positive for all real values of variables, prove that the discriminant of $U - \lambda V$ has all its roots real.

Using the convention that when a suffix occurs twice in a term, such term is to be summed for all values of the suffix from 1 to n, we write $U \equiv a_{\alpha\beta}x_\alpha x_\beta$, $V \equiv b_{\alpha\beta}x_\alpha x_\beta$. If $\Delta \equiv (a_{\alpha\beta} - \lambda b_{\alpha\beta}) = 0$, we can find values of $x_1', x_2', \ldots x_n'$, so that $a_{\alpha\beta}x_\beta' = \lambda b_{\alpha\beta}x_\beta'$, for if all the first minors of $\Delta = 0$, two of $x_1', x_2', \ldots x_n'$ may be taken arbitrarily, because any values which satisfy $n - 2$ of the equations $a_{\alpha\beta}x_\beta' = \lambda b_{\alpha\beta}x_\beta'$ satisfy the remaining two. This may be seen by taking $n - 1$

of them, multiplying by the $n-1$ minors of the second order formed by the coefficients of any $n-2$ variables and adding. The result $\equiv 0$, as the coefficient of any variable is either a minor of the first order or a determinant with two columns identical. Thus unless every minor of the second order vanishes there are at least two linear identical equations connecting $n-2$ of the equations of which the coefficients of $n-2$ variables form a minor not $= 0$, with the other two. Similarly if all the minors of the third order vanish, three of the variables may be taken arbitrarily, as there are at least three linear equations connecting $n-3$ of the equations of which the coefficients of $n-3$ variables form a minor of the third order not $= 0$, with the other three. And so on generally. Hence we can always get values of $x_1', x_2', \ldots x_n'$ which do not all vanish to satisfy $a_{\alpha\beta}x_\beta' = \lambda b_{\alpha\beta}x_\beta'$, if λ is a root of $(a_{\alpha\beta} - \lambda b_{\alpha\beta}) = 0$.

If $\mu + i\nu$ is a root of $(a_{\alpha\beta} - \lambda b_{\alpha\beta}) = 0$, and the corresponding values of $x_\alpha' = \xi_\alpha + i\eta_\alpha$ which do not all vanish are substituted in $a_{\alpha\beta}x_\beta' = (\mu + i\nu)\, b_{\alpha\beta}x_\beta'$, we get

$$a_{\alpha\beta}\,(\xi_\beta + i\eta_\beta) = (\mu + i\nu)\, b_{\alpha\beta}\,(\xi_\beta + i\eta_\beta),$$

and therefore, as $a_{\alpha\beta}$, $b_{\alpha\beta}$ are real

$$a_{\alpha\beta}\xi_\beta = \mu b_{\alpha\beta}\xi_\beta - \nu b_{\alpha\beta}\eta_b$$
$$b_{\alpha\beta}\eta_\beta = \mu b_{\alpha\beta}\eta_b + \nu b_{\alpha\beta}\xi_\beta.$$

Hence multiply the first by η_α and the second by ξ_α and subtracting and summing, we have

$$\nu\{b_{\alpha\beta}\eta_\alpha\eta_\beta + b_{\alpha\beta}\xi_\alpha\xi_\beta\} = 0.$$

But $b_{\alpha\beta}\eta_\alpha\eta_b$ and $b_{\alpha\beta}\xi_\alpha\xi_\beta$ are by hypothesis each positive and do not vanish unless all the ξ's and all the η's $= 0$, and this latter is not the case as we saw, $\therefore\ \nu = 0$, and $\therefore$ all the roots of Δ are real.

18. If U, V are two homogeneous functions of the second degree in n variables with real coefficients, and V is positive for all real values of the variables, prove that U and V may be expressed as sums of squares of the same n real linear functions, with real coefficients.

Transform $U \equiv a_{\alpha\beta}x_\alpha x_\beta$ and $V \equiv b_{\alpha\beta}x_\alpha x_\beta$ by the linear substitution $x_\alpha = x_{\alpha\beta}\, x_\beta'$, getting

$$U = a_{\alpha\beta}x_{\alpha\gamma}x'_\gamma x_{\beta\delta}x_\delta' = a_{\alpha\beta}x_{\alpha1}x_{\beta1}x_1'^2 + 2a_{\alpha\beta}x_{\beta1}x_{\alpha p}x_1'x_p' + a_{pq}x_{pr}x_r'x_{qs}x_s', \quad (p,\, q,\, r,\, s = 2 \text{ to } n).$$

$$V = b_{\alpha\beta}x_{\alpha\gamma}x_\gamma'x_{\beta\delta}x_\delta' = b_{\alpha\beta}x_{\alpha1}x_{\beta1}x_1'^2 + 2b_{\alpha\beta}x_{\beta1}x_{\alpha p}x_1'x_p' + b_{pq}x_{pr}x_{qs}x_r'x_s'.$$

Now λ_1 being any root of $(a_{\alpha\beta} - \lambda b_{\alpha\beta}) = 0$, by the last example λ_1 is real and we can find real values of $x_{11}, x_{21}, x_{31}, \ldots x_{n1}$ to satisfy $a_{\alpha\beta}x_{\beta1} = \lambda_1 b_{\alpha\beta}x_{\beta1}$. Assume as well arbitrary real values for all the other coefficients in the substitution, but so that the modulus does not vanish.

Multiply $a_{\alpha\beta}x_{\beta1} = \lambda_1 b_{\alpha\beta}x_{\beta1}$ by $x_{\alpha1}$ and summing, and by $x_{\alpha p}$ and summing, we get $x_{\alpha\beta}x_{\alpha1}x_{\beta1} = \lambda_1 b_{\alpha\beta}x_{\alpha1}x_{\beta1}$, $a_{\alpha\beta}x_{\beta1}x_{\alpha p} = \lambda_1 b_{\alpha\beta}x_{\beta1}x_{\beta p}$. Therefore the coefficients of $x_1'^2$ and $x_1'x_p'$ in U equal λ_1 times the corresponding coefficients in V. Also noting that $b_{\alpha\beta}x'_{\alpha1}b'_{\beta1}$ is positive and does not vanish, and denoting it by k_1^2, we get, if we put $X_1 = k_1 x_1' + \frac{1}{k_1} b_{\alpha\beta} x_{\beta 1} x_{\alpha p} x_p'$, $U = \lambda_1 X_1^2 + U'$, $V = X_1^2 + V'$, where U', V' are functions of $x_2' x_3' \ldots x_n'$ and V' is still positive for all values of the variables, since $U' = U$ for any values of $x_2' x_3' \ldots x_n'$, if x_1' is found from $X_1 = 0$. We now proceed similarly with U', V', having reduced the variables to $n - 1$, and so step by step we establish the desired result.

If finally $X_\alpha = l_{\alpha\beta} x_\beta$, and $(l_{\alpha\beta}) = M$, we have the discriminant Δ of $U - \lambda V$

$$= M^2 (\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_n),$$

so that $\lambda_1, \lambda_2, \ldots \lambda_n$ are the roots of $\Delta = 0$.

We may further note that if Δ has two roots $= \lambda_1$, the discriminant Δ' of the final form $U - \lambda V$, having terms only in the diagonal, is such that for $\lambda = \lambda_1$ all minors of first order vanish. Hence as Δ is obtained from Δ' by multiplying by a determinant $A = M^2$, and so any first minor of Δ is the product of two arrays formed by $n - 1$ rows of A and $n - 1$ rows of Δ', and so is a linear function of first minors of Δ', and so vanishes for $\lambda = \lambda_1$. Similarly if $\lambda = \lambda_1$ is a triple root of $\Delta = 0$, when $\lambda = \lambda_1$, all the minors of the second order of $\Delta = 0$, and so on generally.

19. Express in general three cubics U, V, W, by means of three cubes. Putting $U = A(x - \alpha y)^3 + B(x - \beta y)^3 + C(x - \gamma y)^3 \equiv Au^3 + Bv^3 + Cw^3$, we have $a_0 = A + B + C$, $-a_1 = A\alpha + B\beta + C\gamma$, $a_2 = A\alpha^2 + B\beta^2 + C\gamma^2$, $-a_3 = A\alpha^3 + B\beta^3 + C\gamma^3$, and so if α, β, γ are roots of $p_0 x^3 + p_1 x^2 y + p_2 x y^2 + p_3 y^3$, $p_0 a_3 - p_1 a_2 + p_2 a_1 - p_3 a_0 = 0$. Similarly putting $V \equiv A'u^3 + B'v^3 + C'w^3$, and $W \equiv A''u^3 + B''v^3 + C''w^3$, we get $p_0 b_3 - p_1 b_2 + p_2 b_1 - p_3 b_0 = 0$, $p_0 c_3 - p_1 c_2 + p_2 c_1 - p_3 c_0$. Hence α, β, γ are roots, and u, v, w factors of

$$K \equiv \begin{vmatrix} x^3 & -x^2 y & x y^2 & -y^3 \\ a_3 & a_2 & a_1 & a_0 \\ b_3 & b_2 & b_1 & b_0 \\ c_3 & c_2 & c_1 & c_0 \end{vmatrix}$$

Finding then $A, B, C, A', B', C', A'', B'', C''$ from the first three of each of the three sets of four equations, we obtain the desired result when α, β, γ are all different, or K of form uvw.

Putting $\beta = \alpha + \epsilon$, $\gamma = \alpha + \epsilon + \eta$, getting A, B, C, and finding the limit when $\epsilon = 0$, we get that if K is of the form $u^2 v$, U, V, W are each of the form $Au^3 + Bu^2 v + Cv^3$. Further, finding the limit when $\eta = 0$, we get that when K is of the form u^3, u is a factor of U, V, W.

If $K \equiv 0$, there is a linear relation between U, V, W.

A similar method may be applied to express in general n quantics of the n^{th} order in terms of n n^{th} powers.

20. Prove that the three roots of a cubic may be expressed as

$$x_1, \quad \theta(x_1), \quad \theta\theta(x_1) = \theta^2(x_1),$$

where

$$\theta(x) = \frac{lx + m}{l'x + m'}, \quad \text{and} \quad \theta^3(x) = x.$$

This follows from Art, 60, Vol. I., or from the fact (cf. Art, 206) that any cubic may be linearly transformed into itself, but it may be proved in a more elementary and satisfactory way by finding l, m, l', m', from the equations

$$l'\beta\gamma - l\beta + m'\gamma - m = 0,$$
$$l'\gamma\alpha - l\gamma + m'\alpha - m = 0,$$
$$l'\alpha\beta - l\alpha + m'\beta - m = 0.$$

Assuming α, β, γ unequal we easily find that we may take $l' = ac - b^2$, $m' - l = ad - bc$, $m' + l = \sqrt{-\frac{1}{3}\Delta}$, $m = c^2 - bd$, and we note that $(lm' - l'm) = -\frac{1}{3}\Delta = (m' + l)^2$.

This example is a particular case of a general theorem of Abel's, viz.: If the m roots of an equation of the m^{th} degree are a, $\theta(a)$, $\theta^2(a)$, . . . $\theta^{m-1}(a)$, where $\theta(x)$ is a rational function such that when the operation θ is repeated m times $\theta^m(x) = x$, then the solution requires only the determination of a primitive root of $x^m - 1 = 0$ and the extraction of the m^{th} root of a known quantity (see under Abelian Equations).

21. Given a binary cubic U and its Hessian H_x, the cubic being satisfied by the ratios $x : y$ and $x' : y'$; prove that

$$\frac{1}{\sqrt{\Delta}} \frac{x'\dfrac{\partial H_x}{\partial x} + y'\dfrac{\partial H_x}{\partial y}}{xy' - x'y}$$

is an absolute constant, Δ being the discriminant of U.

This expression is absolutely unchanged by linear transformation, since

$$H_{X,\,Y} = M^2 H_{x,\,y}, \quad \Delta' = M^6\Delta,$$

and

$$\begin{vmatrix} X & Y \\ X' & Y' \end{vmatrix} = \frac{1}{M}\begin{vmatrix} x & y \\ x' & y' \end{vmatrix}, \qquad X'\frac{\partial}{\partial X} + Y'\frac{\partial}{\partial Y} = x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}.$$

Reducing U to the sum of two cubes by a linear transformation whose modulus $= 1$, the constant may be easily shown to be $\dfrac{1}{\sqrt{-3}}$. This is another form of the homographic relation of Art. 60.

22. Prove that a rational homographic relation in terms of the coefficients connects any two rational functions of the same root of a cubic equation; but that the relation is not rational when the roots are different.

23. Transform the quartic

$$(a, b, c, d, e)\ (x, 1)^4$$

into one whose invariant I shall vanish.

Assuming $\qquad y = x^2 + 2\eta x + \zeta,$

and making the invariant I of the transformed equation vanish, we have

$$\Sigma(\rho_2 - \rho_3)^2(\phi - \rho_1)^2 = 0, \tag{1}$$

where ϕ is a known quadratic function of η, not involving ζ.

Expanding (1), we have

$$I\phi^2 - 3J\phi + \frac{I^2}{12} = 0,$$

which determines ϕ, and consequently η, by means of a quadratic equation; and ζ may have any value.

By a similar transformation J can be made to vanish.

24. Prove that the most general rational transformation of a quartic $f(x)$ may be reduced to the transformation

$$y = \frac{P}{p - x} + \frac{Q}{q - x}.$$

When $P = Rf(p)f'(q)$, and $Q = - Rf(q)f'(p)$, show that the second term of the transformed quartic is absent.

25. Prove that the transformation

$$y = \frac{ax^2 + 2\beta x + \gamma}{a_1 x^2 + 2\beta_1 x + \gamma_1}$$

may be accomplished by the three successive transformations—(1) a homographic transformation; (2) a transformation of the roots into their squares; (3) a homographic transformation.

26. If p be any integer, prove that

$$\frac{(x_1{}^p - x_2{}^p)(x_3{}^p - x_4{}^p)}{(x_1 - x_2)(x_3 - x_4)} = \Sigma_0 + (x_1x_2 + x_3x_4)\,\Sigma_1,$$

where Σ_0 and Σ_1 are symmetric functions of x_1, x_2, x_3, x_4; prove also that

$$\frac{(\phi(x_1) - \phi(x_2))(\phi(x_3) - \phi(x_4))}{(\psi(x_1) - \psi(x_2))(\psi(x_3) - \psi(x_4))} = \frac{\Sigma_0 + \Sigma_1(x_1x_2 + x_3x_4)}{\Sigma_0' + \Sigma_1'(x_1x_2 + x_3x_4)},$$

where $\Sigma_0, \Sigma_1, \Sigma_0', \Sigma_1'$ are symmetric functions of x_1, x_2, x_3, x_4. (See Art. 198.)

27. If $\phi(x, y)$ and $\psi(x, y)$ be two covariants of the binary form

$$U \equiv (a_0, a_1, a_2, \ldots a_n)(x, y)^n$$

of the degrees p and q, respectively; and if

$$\phi\left(xX - \frac{1}{q}\frac{\partial\psi}{\partial y}Y,\quad yX + \frac{1}{q}\frac{\partial\psi}{\partial x}Y\right)$$

be expanded in the form

$$(V_0, V_1, V_2, \ldots V_p)(X, Y)^p;$$

prove that $V_0, V_1, V_2, \ldots V_p$ are covariants of U. (Hermite.)

Expanding, the coefficient of $X^{p-j}Y^j$ is

$$\frac{(-1)^j}{1 \cdot 2 \cdot 3 \ldots j}\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)^j\phi.$$

The modulus of this transformation of ϕ is $\psi(x, y)$.

28. When in the preceding example $n = 4$, and $\phi(x, y)$ and $\psi(x, y)$ are replaced by U, find the values of V_0, V_1, V_2, V_3, V_4.

Ans. $(U, 0, H_x, G_x, IU^2 - 3H_x^2)(X, Y)^4$.

29. Prove for two cubics U and V

$$Q^2 = 16 \begin{vmatrix} I_{11} & I_{21} & I_{31} \\ I_{12} & I_{22} & I_{32} \\ I_{13} & I_{23} & I_{33} \end{vmatrix},$$

where I_{11}, I_{12}, &c., are the invariants of the three Hessians, and Q has the same signification as in Art. 192.

30. Eliminate x' from the equations

$$z = (a_0x' + a_1)x + (a_0x'^2 + 3a_1x' + 2a_2)y, \quad (a_0, a_1, a_2, a_3)(x', 1)^3 = 0,$$

Ans. $z^3 + 3H_{x,y}z + G_{x,y} = 0$.

31. Transform the quadric $(a, b, c, f, g, h)(x, y, z)^2$ to X, Y, Z, where

$$X = \alpha_1x + \beta_1y + \gamma_1z, \quad Y = \alpha_2x + \beta_2y + \gamma_2z, \quad Z = \alpha_3x + \beta_3y + \gamma_3z.$$

$$\frac{1}{M_2}\begin{vmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & X \\ \Pi_{21} & \Pi_{22} & \Pi_{23} & Y \\ \Pi_{31} & \Pi_{32} & \Pi_{33} & Z \\ X & Y & Z & 0 \end{vmatrix},$$

where

$$\Pi_{ij} \equiv A\alpha_i\alpha_j + B\beta_i\beta_j + C\gamma_i\gamma_j + F(\beta_i\gamma_j + \beta_j\gamma_i) + G(\gamma_i\alpha_j + \alpha_i\gamma_j) + H(\alpha_i\beta_j + \alpha_j\beta_i),$$

and A, B, C, F, G, H are the coefficients of the tangential form of

$$(a, b, c, f, g, h)(x, y, z)^2.$$

32. Prove that the quartic $(a, b, c, d, e)(x, y)^4$ may be transformed into

$$k\eta(4\xi^3 - I\xi\eta^2 + J\eta^3)$$

by the substitution

$$\xi = lx + my, \quad \eta \equiv x - \delta y,$$

where $\alpha, \beta, \gamma, \delta$ are the roots, and

$$12l = -a\Sigma(\alpha - \delta)(\beta - \delta), \quad 12m = a\Sigma\alpha(\beta - \delta)(\gamma - \delta),$$

in which the summation is with respect to α, β, γ, and k is a function of $\alpha, \beta, \gamma, \delta$.

33. When U_x is a quartic, and H_x its Hessian, prove that the factors of $U_xH_y - U_yH_x$ are $x - y$ and the three quadratic factors of G_x (Art. 183) when xy replaces x^2, and $x + y$ replaces $2x$.

34. Prove that all quartic covariants of U_x whose roots are rational functions of the roots of U_x are included in the formula

$$(\rho^4 + \tfrac{1}{2}I\rho^2 - 2J\rho + \tfrac{1}{16}I^2)U_x - (4\rho^3 - I\rho + J)H_x. \quad \text{(Mr. Russell.)}$$

How is this example connected with the preceding?

35. Prove that $\frac{x-\alpha}{\delta-\alpha}+\frac{x-\beta}{\delta-\beta}+\frac{x-\gamma}{\delta-\gamma}$ is a factor of $I^2U_x - 16JH_x$,

where $$U_x \equiv (x-\alpha)(x-\beta)(x-\gamma)(x-\delta).$$

36. If U_x and U'_ξ be two quartics which have the same absolute invariant, prove that

$$IJ'H_xU'_\xi - I'JH'_\xi U_x$$

may be resolved into four factors of the form

$$Ax\xi + Bx + C\xi + D. \qquad \text{(Mr. Russell.)}$$

37. If the leading coefficient of a covariant involves the coefficients of several quantics in the orders $\varpi_1, \varpi_2, \ldots \varpi_r$ and weights $\kappa_1, \kappa_2, \ldots \kappa_r$, the degree of the covariant is

$$n_1\varpi_1 + n_2\varpi_2 + \ldots + n_r\varpi_r - 2(\kappa_1 + \kappa_2 + \ldots + \kappa_r). \qquad \text{(Art. 166.)}$$

38. If for every difference $a_p - a_q$, in the formation of a seminvariant ϕ of an equation $U=0$, we substitute

$$U\frac{(a_p - a_q)}{(x-a_p)(x-a_q)},$$

prove that the result is the product of the covariant whose leader is ϕ by $U^{\kappa-\varpi}$, where ϖ is the order and κ the weight of ϕ.

39. When U is a quintic, what are the invariants of the quartic emanant

$$\left(x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}\right)^4 U\,?$$

Ans. The quadric and cubic covariants I_x' and J_x'.

40. Give the relation connecting the covariants H_x, G_x, I_x, J_x, of any quantic U. *Ans.* $-G_x^2 = 4H_x^3 - U^2H_xI_x + U^3J_x$.

41. Show how to transform a quantic of an odd order so that all the new coefficients shall be invariants.

Ans. Take two linear covariants for the new X and Y.

42. Find the relation which connects the coefficients of two quartics (if any) when their roots are connected by the relation

$$\begin{vmatrix} 1 & \alpha & \alpha' & \alpha\alpha' \\ 1 & \beta & \beta' & \beta\beta' \\ 1 & \gamma & \gamma' & \gamma\gamma' \\ 1 & \delta & \delta' & \delta\delta' \end{vmatrix}.$$

Ans. $I^3J'^2 - I'^3J^2 = 0$.

(Cf. Exs. 13, 14, p. 119, Vol. I.)

43. Transform a cubic U into its cubic covariant G_x by linear transformation. Making the transformation given by the equation

$$x' \frac{\partial Hx}{\partial x} + y' \frac{\partial Hx}{\partial y} = 0, \quad \text{the result is } \Delta G_x'.$$

44. Transform a quartic U into itself by linear transformation.

If $$U \equiv A(x^4 + y^4) + 2Bx^2y^2,$$

the quadratic factors of the covariant G_x are xy, $x^2 + y^2$, $x^2 - y^2$; now, making the transformation determined by the equation $x'\dfrac{\partial \phi}{\partial x} + y'\dfrac{\partial \phi}{\partial y} = 0$, where ϕ is any one of these three factors, U is transformed into U'.

45. Prove that the quadratic factors u, v, w of G_x expressed in terms of the roots are unchanged when for x, α, β, γ, δ their reciprocals are substituted and fractions removed by the multiplier $(-1)\, x^2\alpha\beta\gamma\delta$.

It appears, therefore, that a_0u, a_0v, a_0w may separately be regarded as covariants, if the rational domain, which before included only the coefficients, be regarded as extended by the adjunction to it of the roots α, β, γ, δ. (See Art. 168.)

46. If three quadratics be mutually harmonic, prove that they may be reduced to the forms

$$AX^2 + CY^2, \quad AX^2 - CY^2, \quad BXY.$$

47. Form for a quintic a seminvariant whose order is 4 and weight 8.

The terms contained in the complete *gradient* * $G_{4,8}$ are as follows :—

$a_0^2a_4^2$, $a_1^2a_2a_4$, $a_0a_1a_3a_4$, $a_0a_2^2a_4$, $a_0a_2a_3^2$, $a_1^2a_3^2$, $a_1a_2^2a_3$, a_2^4, $a_0^2a_3a_5$, $a_0a_1a_2a_5$, $a_1^3a_5$.

Operating with D, and making $DG_{4,8} \equiv 0$, we find seminvariants of the type

$$lS + mI^2,$$

where I has the usual meaning, and

$$S = a_0^2a_4^2 - 3a_0a_1a_3a_4 - 4a_0a_2^2a_4 + 4a_0a_2a_3^2 + 5a_1^2a_2a_4 + 2a_1^2a_3^2 - 8a_1a_2^2a_3 + 3a_2^4 - a_5(a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3).$$

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* The term "gradient" is used to signify the sum, with arbitrary multipliers, of all possible terms of any assigned order and weight.

SECTION IV.—GEOMETRICAL TRANSFORMATIONS.*

212. **Transformation of Binary to Ternary Forms.** —We think it desirable, before closing the present chapter, to give a brief account of a simple transformation from a binary to a ternary system of variables, whereby a geometrical interpretation may be given to several of the results contained in the preceding chapters. The applications which follow will be sufficient to explain this mode of transformation ; and will enable the student acquainted with the principles of analytic geometry to trace further the analogy which exists between the two systems.

Denoting the original variables, i.e. the variables of the binary system, by x, y, we propose to transform to a ternary system by the substitutions

$$X = x^2, \quad Y = 2xy, \quad Z = y^2,$$

a form to which the general quadratic transformation may be brought † by a linear transformation of the ternary variables.

For example, taking the simple case of a quadratic whose roots are α, β, viz.,

$$x^2 - (\alpha + \beta)\,xy + \alpha\beta y^2 = 0,$$

and transforming, we obtain

$$X - \tfrac{1}{2}(\alpha + \beta)\,Y + \alpha\beta Z = 0. \qquad (1)$$

We have also the equation

$$4XZ - Y^2 = 0.$$

This is the equation of a conic, which we uniformly call K, and (1) is plainly the equation of a chord of this conic joining the *points* α and β, the point determined by the equations

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* See *Quarterly Journal of Mathematics*, vol. x., p. 211, 1869.

† If $X' = ax^2 + 2bxy + cy^2$, $Y' = a_1x^2 + 2b_1xy + c_1y^2$, $Z' = a_2x^2 + 2b_2xy + c_2y^2$, solving for x^2, xy, y^2, on the assumption $(a_1, b_2, c_3) \neq 0$, as if it does all points lie on a line, we obtain $x^2 = (AX' + A_1Y' + A_2Z')/(a_1b_2c_3) = X$, with similar values for Y, Z.

$$\frac{X}{\phi^2} = \frac{Y}{2\phi} = Z, \text{ where } \phi = \frac{x}{y},$$

being referred to as the point ϕ on the conic K.

When $\alpha = \beta$ the quadratic becomes $(x - \alpha y)^2$, i.e. the square of a factor of the first degree; also (1) reduces to $X - \alpha Y + \alpha^2 Z = 0$, which is plainly the equation of the tangent at the point α to the conic K; whence the line corresponding to a quadratic with distinct roots is a chord of the conic K, this line becoming a tangent when the roots are equal.

In further illustration of this method, we consider the binary sextic and quintic, so as to show how the transformation is presented differently according as the degree of the quantic is even or odd. In the former case we have

$$u \equiv (x - \alpha_1 y)(x - \alpha_2 y)(x - \alpha_3 y)(x - \alpha_4 y)(x - \alpha_5 y)(x - \alpha_6 y),$$

which becomes by transformation

$$c_{12}c_{34}c_{56}, \quad c_{12}c_{35}c_{46}, \quad c_{12}c_{36}c_{45},$$

or some other of the fifteen similar products of chords, where $c_{12} \equiv X - \frac{1}{2}(\alpha_1 + \alpha_2)Y + \alpha_1\alpha_2 Z$ is the chord 1, 2; and c_{34}, c_{56}, &c., have a like signification. In the second case, viz., when the degree of the binary quantic is odd, we must square u before making the transformation. Thus if u represents the product of the first five factors written above, u^2 becomes when transformed $t_1t_2t_3t_4t_5$, where $t_1 \equiv X - \alpha_1 Y + \alpha_1^2 Z$ is the tangent to K at the point α_1, and t_2, t_3, &c., have a like signification. As, however, in such transformations we may replace $4x^2y^2$ by $4XZ$ or by $Y^2 \equiv 4XZ - K$, they may be effected in a variety of ways, but we see that two such transformations differ by an expression of which K is a factor.

213. The Quadratic and Systems of Quadratics.— The only invariant that a quadratic has is its discriminant; and this is also an invariant in the ternary system, its vanishing being the condition that the line corresponding to the quadratic should touch the conic K. We now consider the system of two quadratics

$$ax^2 + 2bxy + cy^2, \quad a'x^2 + 2b'xy + c'y^2,$$

which we call l and m.

When transformed they become two lines

$$L \equiv aX + bY + cZ, \quad M \equiv a'X + b'Y + c'Z.$$

Now the condition that the line whose equation is $\lambda L + \mu M = 0$ should touch the conic K is

$$\lambda^2(ac - b^2) + \lambda\mu\,(ac' + a'c - 2bb') + \mu^2(a'c' - b'^2) = 0. \qquad (2)$$

All the coefficients of this equation are invariants in both systems: we have already seen that this is true of the first and last coefficients; and the intermediate coefficient which is the harmonic invariant of the binary system is an invariant in the ternary system also, its vanishing expressing the condition that the lines L, M should be conjugate with regard to the conic K. This equation determines the tangents which can be drawn through the point of intersection of L and M to the conic K. When this point is on the conic, the tangents coincide, and the discriminant of the quadratic vanishes. Whence we obtain geometrically the following form for the resultant of two quadratics:—

$$R = 4\,(ac - b^2)\,(a'c' - b'^2) - (ac' + a'c - 2bb')^2\,;$$

for if L, M, and K have a common point, the original quadratics must have a common root, and the condition is in each case the same.

Again, the pairs of points or lines given by the equation $\lambda L + \mu M = 0$ form a system in involution (cf. Art. 190), the double points or lines being determined by the equation (2); and in the ternary system the corresponding pencil of lines passing through a fixed point determines on a conic a system of points in involution, the double points being the points of contact of tangents drawn to the conic from the fixed point.

If we consider next the three quadratics

$$a_1x^2 + 2b_1xy + c_1y^2, \quad a_2x^2 + 2b_2xy + c_2y^2, \quad a_3x^2 + 2b_3xy + c_3y^2,$$

it is seen that the determinant $(a_1b_2c_3)$ is an invariant in both systems, its vanishing being the condition in the binary system that the quadratics should determine an involution (Ex. 16, p. 208), and in the ternary system that the three corresponding lines should meet in a point.

As a final illustration, we consider a system of three quadratics connected in pairs by the harmonic relations

$$a_1c_2 + a_2c_1 - 2b_1b_2 = 0, \text{ \&c.}$$

Transforming the quadratics, we obtain three lines L, M, N, which form a self-conjugate triangle with regard to the conic K. The theorem relating to three mutually harmonic quadratics, viz., that their squares are connected by an identical linear relation (see Ex. 6, p. 136), is suggested by a well-known property of conics; for K may be expressed in terms of L, M, N in the form

$$K \equiv L^2 + M^2 + N^2;$$

whence, restoring the original variables x, y, K vanishes identically, and L, M, N become the original quadratics, each divided by a factor which may be seen to be the square root of its discriminant (see (1), Ex. 6, p. 136; also Ex. 18, p. 209).

214. **The Quartic and its Covariants treated geometrically.**—It will appear from the investigations in the succeeding Articles that in applying the transformation now under consideration to the quartic $u \equiv (a, b, c, d, e)(x, y)^4$, the term $6cx^2y^2$ should be replaced by $2cXZ + cY^2$, so that the quartic will be replaced by the two following conics:—

$$U \equiv aX^2 + cY^2 + eZ^2 + 2dYZ + 2cZX + 2bXY,$$
$$K \equiv 4ZX - Y^2,$$

the form of U here selected being connected with K by an invariant relation. The invariants of U and K are invariants of the original binary form, for the discriminant of $U + \rho K$ is

$$4\rho^3 - I\rho + J,$$

and hence the invariants of the ternary system are

$$\Delta' = 4, \quad \Theta' = 0, \quad \Theta = -I, \quad \Delta = J;$$

where I and J are the invariants of the quartic, the discriminant of $U + \rho K$ being written as usual under the form

$$\Delta + \rho\Theta + \rho^2\Theta' + \rho^3\Delta'.$$

Let the conics U and K intersect in the points A, B, C, D, these points being determined by the equations

$$\frac{X}{\phi^2} = \frac{Y}{2\phi} = Z,$$

when ϕ has the four values α, β, γ, δ, the roots of the binary quartic; and let the points of intersection of the common chords BC, AD; CA, BD; AB, CD be E, F, G, respectively, where EFG is the triangle self-conjugate with regard to both conics. Now, denoting by $(\alpha\beta) = 0$ the equation of the line AB, and using a similar notation for the remaining chords, we have by the theory of conics

$$U + \rho_1 K = (\beta\gamma)(\alpha\delta), \quad U + \rho_2 K = (\gamma\alpha)(\beta\delta), \quad U + \rho_3 K = (\alpha\beta)(\gamma\delta),$$

where ρ_1, ρ_2, ρ_3 are the roots of the equation $4\rho^3 - I\rho + J = 0$.

On restoring the original variables x, y in these equations, K vanishes identically; and we have u resolved into a pair of quadratic factors in three different ways, depending on the solution of the reducing cubic of the quartic. Whence it appears that the resolution of a quartic into its pairs of quadratic factors, and the determination of the pairs of lines which pass through the four intersections of two conics, are identical problems, each depending on the solution of the same cubic equation.

We now proceed to show that the sides of the common self-conjugate triangle of U, K correspond to the quadratic factors of the sextic covariant in the binary system. Since the side FG

is the polar of E, the coordinates X', Y', Z' of E are found by solving the equations $(\beta\gamma) = 0$, $(\alpha\delta) = 0$; we have, therefore,

$$\frac{X'}{\beta\gamma(\alpha+\delta) - \alpha\delta(\beta+\gamma)} = \frac{Y'}{2(\beta\gamma - \alpha\delta)} = \frac{Z'}{\beta+\gamma-\alpha+\delta},$$

and, substituting for X', Y', Z' the values thus determined in the polar of E, viz.,

$$XZ' - \frac{YY'}{2} + X'Z = 0,$$

we express this equation in the form

$$(\beta+\gamma-\alpha-\delta)X - (\beta\gamma - \alpha\delta)Y + (\beta\gamma(\alpha+\delta) - \alpha\delta(\beta+\gamma))Z = 0.$$

On restoring the original variables x, y, this is seen to be one of the quadratic factors of the sextic covariant (Art. 183). It is therefore proved that the points where FG meets K are determined by the quadratic equation

$$(\beta+\gamma-\alpha-\delta)\phi^2 - 2(\beta\gamma-\alpha\delta)\phi + \beta\gamma(\alpha+\delta) - \alpha\delta(\beta+\gamma) = 0;$$

and consequently the six points on K which correspond to the roots of the sextic covariant are the points where this conic meets the sides of the common self-conjugate triangle of U and K.

To determine the points on K which correspond to the roots of the Hessian, we calculate for the conics U and K the covariant conic F (Salmon's *Conic Sections*, Art. 378); thus finding

$$-\tfrac{1}{4}F \equiv (ac-b^2)X^2 + (bd-c^2)Y^2 + (ce-d^2)Z^2 + (be-cd)YZ$$
$$+ (ae - 2bd + c^2)ZX + (ad-bc)XY;$$

and on restoring the original variables, we have

$$H(x,y)^4 = -\tfrac{1}{4}f;$$

also, since the conic F intersects U and K in the points of contact of their common tangents, we see that the points on K corresponding to the roots of the Hessian are the points so determined. Conversely, transforming the Hessian $U_{11}U_{22} - U_{12}^2$ to the ternary variables, it becomes

$$(aX+bY+cZ)(cX+dY+eZ) - (bX+cY+dZ)^2 \equiv -\tfrac{1}{4}F,$$

which is the envelope of the polar $X'U_1 + Y'U_2 + Z'U_3$ for all points on K. This determination of F is due to the invariant Θ' of U and K vanishing.

215. We now give some general transformations from the binary system to the ternary, which will be useful in comparing the concomitants in both systems.

(1). *Linear transformation of both systems.*

If the binary variables be linearly transformed, the new variables expressed in terms of the old being

$$x' = \lambda x + \mu y, \quad y' = \lambda' x + \mu' y,$$

the new ternary variables will be expressed in terms of the old as follows :—

$$X' = \lambda^2 X + \lambda\mu Y + \mu^2 Z,$$
$$Y' = 2\lambda\lambda' X + (\lambda\mu' + \lambda'\mu) Y + 2\mu\mu' Z,$$
$$Z' = \lambda'^2 X + \lambda'\mu' Y + \mu'^2 Z;$$

and, consequently,

$$4Z'X' - Y'^2 = (\lambda\mu' - \lambda'\mu)^2 (4ZX - Y^2),$$

showing that *the form of the fixed conic is unaltered* by the above *particular* linear transformation of X, Y, Z, which conversely leads to the *general* linear transformation of the primitive binary variables. The modulus of this ternary transformation is $(\lambda\mu' - \lambda'\mu)^3$ (see Ex. 4, p. 89).

(2). *Transformation of Partial Differential Coefficients.*

If $u\,(x, y)$ becomes U by the substitution of Art. 212, we have

$$\frac{\partial u}{\partial x} = 2x\frac{\partial U}{\partial X} + 2y\frac{\partial U}{\partial Y},$$

and therefore

$$\frac{\partial^2 u}{\partial x^2} = 2\frac{\partial U}{\partial X} + 4\left(X\frac{\partial^2 U}{\partial X^2} + Y\frac{\partial^2 U}{\partial X\partial Y} + Z\frac{\partial^2 U}{\partial X\partial Z}\right) - 4Z\left(\frac{\partial^2 U}{\partial Z\partial X} - \frac{\partial^2 U}{\partial Y^2}\right)$$
$$= 4\frac{\partial}{\partial X}\left(X\frac{\partial U}{\partial X} + Y\frac{\partial U}{\partial Y} + Z\frac{\partial U}{\partial Z}\right) - 2\frac{\partial U}{\partial X} - 4Z\Pi(U),$$

where Π is used to denote the operation $\dfrac{\partial^2}{\partial Z\partial X} - \dfrac{\partial^2}{\partial Y^2}$.

Hence, the degree of u being n, and therefore of U being $\frac{1}{2}n$, we have

$$\frac{\partial^2 u}{\partial x^2} = 2(n-1)\frac{\partial U}{\partial X} - 4Z\Pi(U), \quad \text{and similarly}$$
$$\frac{\partial^2 u}{\partial x \partial y} = 2(n-1)\frac{\partial U}{\partial Y} + 2Y\Pi(U),$$
$$\frac{\partial^2 u}{\partial y^2} = 2(n-1)\frac{\partial U}{\partial Z} - 4X\Pi(U).$$

If the transformation be such that $\Pi(U)$ vanishes identically, we have, for the transformation of the second differential coefficients, the following simple values :—

$$\frac{\partial^2 u}{\partial x^2} = 2(n-1)\frac{\partial U}{\partial X}, \quad \frac{\partial^2 u}{\partial x \partial y} = 2(n-1)\frac{\partial U}{\partial Y}, \quad \frac{\partial^2 u}{\partial y^2} = 2(n-1)\frac{\partial U}{\partial Z}.$$

From these values we find easily

$$\frac{1}{2}\left(x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}\right)^2 u = (n-1)\left(X'\frac{\partial U}{\partial X} + Y'\frac{\partial U}{\partial Y} + Z'\frac{\partial U}{\partial Z}\right),$$

showing that *the second emanant* (Art. 174) *in the binary system is transformed into the first polar in the ternary system.*

To return to the consideration of the connexion between the binary and ternary variables, and to show that the fundamental properties of the quantics correspond in the two systems :—

Suppose we had three equations in general connecting the variables

$$X = \phi_1(x, y), \quad Y = \phi_2(x, y), \quad Z = \phi_3(x, y),$$

which we saw may be reduced to the form

$$X = x^2, \quad Y = 2xy, \quad Z = y^2,$$

we may by elimination of xy obtain an equation in X, Y, Z, and we have another relation between X, Y, Z from the transformed binary equation, the roots of which will plainly be represented geometrically by the intersection of these two curves.

The analogy between the two systems of representation, being points on a line and on a conic, will be apparent to a student of analytic geometry, so we give an example where the analogy is apparent. We show how a double root of u leads to a quintuple root of the sextic covariant (Ex. 2, p. 236), for two of the sides of the polar triangle of U and K touch the conic at the vertex of the triangle, and the pole of the third side is a point on the tangent.

(3). *Transformation of the Jacobian.*

The Jacobian of any binary system u, v, is transformed into the Jacobian of U, V, K, where U, V are the transformed values of u, v, the transformations being not necessarily such that $\Pi(U) = 0$, $\Pi(V) = 0$. For we have

$$J(u, v) = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\[2ex] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \frac{1}{(n-1)(n'-1)} \begin{vmatrix} ax + by & bx + cy \\ a'x + b'y & b'x + c'y \end{vmatrix},$$

n and n' being the degrees of u and v, respectively, and a, b, c being used to denote the second differential coefficients; whence we have

$$J(u, v) = \frac{1}{(n-1)(n'-1)} \begin{vmatrix} a & b & c \\ a' & b' & c' \\ y^2 & -xy & x^2 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial U}{\partial X} & \dfrac{\partial U}{\partial Y} & \dfrac{\partial U}{\partial Z} \\[2ex] \dfrac{\partial V}{\partial X} & \dfrac{\partial V}{\partial Y} & \dfrac{\partial V}{\partial Z} \\[2ex] \dfrac{\partial K}{\partial X} & \dfrac{\partial K}{\partial Y} & \dfrac{\partial K}{\partial Z} \end{vmatrix}$$

the last determinant being obtained from the preceding by using the transformation in (2), adding the last row multiplied by $4\Pi(U)$ to the first, and the last row multiplied by $4\Pi(V)$ to the second.

(4). *The Hessian and other concomitants.*

For the transformation of the Hessian we have

$$n^2(n-1)^2 H(u) = \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x\,\partial y}\right)^2$$

$$= 4(n-1)^2 \left\{\frac{\partial U}{\partial X}\frac{\partial U}{\partial Z} - \left(\frac{\partial U}{\partial Y}\right)^2\right\}, \text{ if } \Pi(U) \equiv 0,$$

which proves that one curve into which the Hessian may be transformed is the locus of the poles with regard to U of tangents to the fixed conic.

The line corresponding to the binary concomitant

$$(xy' - x'y)^2 \text{ is } XZ' - \tfrac{1}{2}YY' + ZX',$$

which is the polar of X', Y', Z' with regard to the fixed conic K.

If the quadratics

$$ax^2 + 2bxy + cy^2, \quad a'x^2 + 2b'xy + c'y^2$$

become, when transformed, the lines L, M, the Jacobian $J(L, M, K)$ determines the polar line of their intersection with regard to the fixed conic K.

When $\Pi(U) \equiv 0$, $\Pi(V) \equiv 0$, the curve corresponding to the covariant $(1, 2)^2\, u_1 v_2$ is plainly

$$\frac{\partial U}{\partial x}\frac{\partial V}{\partial z} + \frac{\partial U}{\partial z}\frac{\partial V}{\partial x} - 2\frac{\partial U}{\partial y}\frac{\partial V}{\partial y},$$

which equated to zero is the condition that the polar lines of a point with regard to U and V should be conjugate with regard to the fixed conic. This covariant may be written under the form $\Pi(UV)$, as $\Pi(U) \equiv 0$ and $\Pi(V) \equiv 0$.

216. When the transformation of Art. 212 is applied to a quantic $f(x, y)$ of even degree $2m$, it is plain that the roots of this quantic will be determined geometrically by the points of intersection of a curve of the m^{th} degree with the fixed conic K. If the degree of the quantic is odd, it must, as already stated, be squared before the transformation is effected; and the roots will then be determined geometrically by the points of contact of the corresponding curve with the conic.

In transforming the quantic $u(x, y)$ we may obtain, as we saw, a number of ternary forms by varying the mode of transformation; also, if U be one of these forms, $U + \phi_{m-2}K$ (in which the coefficients of ϕ_{m-2} are arbitrary now) would be a transformation of $u(x, y)$, since this form would, on restoring the original variables, return to the quantic $u(x, y)$. Moreover, system U, V, W, &c., combined with $K \equiv 4ZX - Y^2$. We denote corresponding sets of variables by x_1^α, x_2^α, and X_1^α, X_2^α, X_3^α, and corresponding differential symbols by d_1^α, d_2^α and D_1^α, D_2^α, D_3^α. We also denote $u\,(x_1^\alpha, x_2^\alpha)$ by u_α, $U\,(X_1^\alpha, X_2^\alpha, X_3^\alpha)$ by U_α, and the invariant differential operators

$$\begin{vmatrix} d_1^\alpha, & d_2^\alpha \\ d_1^\beta, & d_2^\beta \end{vmatrix} \text{ by } (\alpha, \beta) \text{ and } \begin{vmatrix} D_1^\alpha, & D_2^\alpha, & D_3^\alpha \\ D_1^\beta, & D_2^\beta, & D_3^\beta \\ D_1^\gamma, & D_2^\gamma, & D_3^\gamma \end{vmatrix} \text{ by } (\alpha, \beta, \gamma).$$

(1) $(\alpha, \beta, \gamma) U_\alpha V_\beta W_\gamma$

$$= \frac{1}{8\,(n_1-1)(n_2-1)(n_3-1)} \begin{vmatrix} d_1^\alpha d_1^\alpha, & d_1^\alpha d_2^\alpha, & d_2^\alpha d_2^\alpha \\ d_1^\beta d_1^\beta, & d_1^\beta d_2^\beta, & d_2^\beta d_2^\beta \\ d_1^\gamma d_1^\gamma, & d_1^\gamma d_2^\gamma, & d_2^\gamma d_2^\gamma \end{vmatrix} u_\alpha v_\beta w_\gamma$$

$$= \frac{1}{8\,(n_1-1)\,(n_2-1)\,(n_3-1)} (\alpha, \beta)(\alpha, \gamma)(\beta, \gamma) u_\alpha v_\beta w_\gamma.$$

where n_1, n_2, n_3 are the degrees of u, v, w, respectively.

(2) $(\alpha, \beta, \gamma)\, U_\alpha V_\beta K_\gamma$

$$= \frac{1}{4(n_1-1)(n_2-1)} \begin{vmatrix} d_1^\alpha d_1^\alpha, & d_1^\alpha d_2^\alpha, & d_2^\alpha d_2^\alpha \\ d_1^\beta d_1^\beta, & d_1^\beta d_2^\beta, & d_2^\beta d_2^\beta \\ 4x_2^\gamma x_2^\gamma, & -4x_1^\gamma x_2^\gamma, & 4x_1^\gamma x_1^\gamma \end{vmatrix} u_\alpha v_\beta$$

$$= \frac{1}{(n_1-1)(n_2-1)} \Delta_{\gamma\alpha} \Delta_{\gamma\beta} (\alpha\beta) u_\alpha v_\beta$$

where $\Delta_{\gamma\alpha} = x_1^\gamma d_1^\alpha + x_2^\gamma d_2^\alpha$.

(3) Similarly, we can easily see that

$$(\alpha, \beta, \gamma) U_\alpha K_\beta K_\gamma = \frac{8}{(n_1-1)} (x_1^\beta x_2^\gamma - x_1^\gamma x_2^\beta) \Delta_{\beta\alpha} \Delta_{\gamma\alpha} u_\alpha.$$

As the operators on the right-hand side of the equations (1), (2), (3) are invariant operators, it follows that the product of any number of symbols of the form (α, β, γ) transforms into the product of invariant operators in the binary system, and so all concomitants of the ternary system are concomitants of the binary.

To see conversely that all concomitants of the binary system are also concomitants of the ternary, we show that after the sets of variables are all made the same, the product of any number of operators of the form $(\alpha\beta)$ is equal to the product of operators of the form $(\alpha\beta\gamma)$ where γ is associated with K_γ, multiplied by a numerical factor.

By (2) $\quad (\alpha, \beta, \epsilon)(\gamma, \delta, \zeta) U_\alpha U_\gamma V_\beta V_\delta K_\epsilon K_\zeta$

$$= \frac{1}{(n_1-1)(n_2-1)} \Delta_{\epsilon\alpha}\Delta_{\epsilon\beta}(\alpha\beta) \frac{1}{(n_1-1)(n_2-1)} \Delta_{\zeta\gamma}\Delta_{\zeta\delta}(\gamma\delta) u_\alpha u_\gamma v_\beta v_\delta.$$

The operators on the right may be taken in any order as there is no differentiation with regard to the variables with affix ϵ or ζ. Taking then $\Delta_{\zeta\delta}$, as there is no subsequent differentiation with regard to the variables with affix δ or ζ, and as finally all sets of variables are to be made the same, in $\Delta_{\zeta\delta}$ ζ may be made equal to δ and therefore $\Delta_{\zeta\delta}$ may be replaced by $n_2 - 1$. Similarly treating $\Delta_{\zeta\gamma}$, $\Delta_{\epsilon\beta}$, $\Delta_{\epsilon\alpha}$, we get that after the sets of variables are finally made the same

$$(\alpha, \beta, \epsilon)(\gamma, \delta, \zeta) U_\alpha U_\gamma V_\beta V_\delta K_\epsilon K_\zeta = (\alpha\beta)(\gamma\delta) u_\alpha u_\gamma v_\beta v_\delta.$$

In the above we have assumed that α is not equal to γ, nor β to δ, and to obtain our desired result we never take $\epsilon = \zeta$. If, then, we take $\alpha = \gamma$,

$$(\alpha\beta\epsilon)(\alpha\delta\zeta) U_\alpha V_\beta V_\delta K_\epsilon K_\zeta$$

$$= \frac{1}{(n_1-3)(n_2-1)} \Delta_{\epsilon\alpha}\Delta_{\epsilon\beta}(\alpha\gamma) \frac{1}{(n_1-1)(n_2-1)} \Delta_{\zeta\alpha}\Delta_{\zeta\delta}(\alpha\delta) u_\alpha v_\beta v_\delta$$

$$= \frac{1}{(n_1-3)(n_1-1)(n_2-1)^2} \cdot \Delta_{\epsilon\alpha}\Delta_{\zeta\alpha}\Delta_{\epsilon\beta}\Delta_{\zeta\delta} \cdot (\alpha\beta)(\alpha\delta) u_\alpha v_\beta v_\delta.$$

As before, $\Delta_{\zeta\delta}$, $\Delta_{\epsilon\beta}$ may be each replaced by $n_2 - 1$, and in $\Delta_{\epsilon\alpha}$, $\Delta_{\zeta\alpha}$ after it is expanded ϵ and ζ may be replaced by α and so the product $\Delta_{\epsilon\alpha}\Delta_{\zeta\alpha}$ may be replaced by $(n_1-2)(n_1-3)$. So, after the operations are completed, and all sets of variables made the same, the right-hand side in the above

$$\frac{n_1-2}{n_1-1}(\alpha\beta)(\alpha\delta) u_\alpha v_\beta v_\delta.$$

It is now clearly seen that the final result of any number of operators of the form $(\alpha\beta)$ gives the same result to a numerical factor as the product of the corresponding operators of form $(\alpha\beta\epsilon)$, where ϵ is associated with K, and no two values of ϵ are equal. Hence all concomitants of the binary system are also concomitants of the ternary.

217. **Combined System of a Quartic and Quadratic.** —Transforming this binary system, we have a ternary system composed of two conics and a line; and for simplicity we shall suppose the conics referred to their common self-conjugate triangle. Denoting, respectively, ternary forms of the quartic and quadratic by U and L, and remembering that we transform so that $\Pi\ (U) \equiv 0$, or so that the result of putting differential symbols in the tangential equation of K and operating on U vanishes identically, or so that the coefficient of ρ in the discriminant of $U + \rho K$ vanishes identically, we have

$$U = aX^2 + bY^2 + cZ^2, \qquad a + b + c = 0,$$
$$K = X^2 + Y^2 + Z^2, \qquad bc + ca + ab = I_2,$$
$$L = \alpha X + \beta Y + \gamma Z, \qquad abc = I_3.$$

To obtain the linear covariants of this system, since α, β, γ are the coordinates of the pole of L with regard to K, the polar of this point with regard to U is $a\alpha X + b\beta Y + c\gamma Z \equiv M$, the first covariant; and treating M in the same way, $a\alpha$, $b\beta$, $c\gamma$ being the coordinates of its pole with regard to K, the polar of this point with regard to U is $a^2\alpha X + b^2\beta Y + c^2\gamma Z \equiv N$, which is a second covariant (see p. 223). We cannot derive any more independent linear covariants in this way, for the next one so derived is

$$a^3\alpha X + b^3\beta Y + c^3\gamma Z = a\,(bc - I_2)\,\alpha X + b(ca - I_2)\,\beta Y + c\,(ab - I_2)\,\gamma Z,$$

which can therefore be expressed in terms of L and M in the form $I_3 L - I_2 M$. But three more linear covariants L', M', N' may be obtained by taking the poles of L, M, N with regard to K, and joining them two and two. This system may be expressed by the Jacobians

$$J(M, N, K), \quad J(N, L, K), \quad J(L, M, K).$$

We have therefore obtained six linear covariants, viz., L, M, N, and L', M', N'; to which all others may be reduced, for example,

$$\begin{aligned} T_n &\equiv a^n\alpha X + b^n\beta Y + c^n\gamma Z \\ &= a^{n-2}(bc - I_2)\,\alpha X + b^{n-2}(ca - I_2)\,\beta Y + c^{n-2}(ab - I_2)\,\gamma Z \\ &= I_3 T_{n-3} - I_2 T_{n-2}; \end{aligned}$$

also

$$b^2c^2\alpha X + c^2a^2\beta Y + a^2b^2\gamma Z = I_2{}^2 L + I_3 M + I_2 N,$$

since

$$bc = a^2 + I_2, \quad ca = b^2 + I_2, \quad ab = c^2 + I_2.$$

Similarly, $b^n c^n \alpha X + c^n a^n \beta Y + a^n b^n \gamma Z$ may be reduced to the form $AL + BM + CN$; and other reductions which present themselves impose no difficulty.

These six linear covariants when transformed give six quadratic covariants in the binary system.

There are six invariants, but only three are special invariants of this system. To obtain them, let the condition that $\lambda L + \mu M + \nu N$ should touch K be

$$D_0\lambda^2 + D_2\mu^2 + D_4\nu^2 + 2D_3\mu\nu + 2D_2\nu\lambda + 2D_1\lambda\mu = 0;$$

whence we obtain five invariants, D_0, D_1, D_2, D_3, D_4, where $D_n \equiv a^n\alpha^2 + b^n\beta^2 + c^n\gamma^2$, three of which only are independent, for

$$\begin{aligned} D_n &= a^{n-2}(bc - I_2)\,\alpha^2 + b^{n-2}(ca - I_2)\,\beta^2 + c^{n-2}(ab - I_2)\,\gamma^2 \\ &= I_3 D_{n-3} - I_2 D_{n-2}; \end{aligned}$$

whence

$$D_3 = I_3 D_0 - I_2 D_1, \quad D_4 = I_3 D_1 - I_2 D_2;$$

and thus we obtain no more than the five invariants I_2, I_3, D_0, D_1, D_2, the two last being special invariants. D_1 vanishes when L and M are conjugate with regard to K, and D_2 when L and N are conjugate with regard to K.

quadratic $L_x, \equiv I_D(u)$; for, by treating these as a combined system, in the manner of Art. 217, we may obtain all the forms of the binary sextic as far as the fourth degree.

Transforming the sextic $u \equiv (a_0, a_1, a_2, a_3, a_4, a_5, a_6)(x, y)^6$, we have the ternary cubic U, such that $\Pi(U) \equiv 0$,

$$U \equiv a_0X^3 + a_3Y^3 + a_6Z^3 + 6a_3XYZ$$
$$+ 3\{a_1X^2Y + a_2X^2Z + a_2Y^2X + a_4Y^2Z + a_4Z^2X + a_5Z^2Y\}.$$

Now, forming the discriminant of

$$\frac{1}{6}\left(X'\frac{\partial}{\partial X} + Y'\frac{\partial}{\partial Y} + Z'\frac{\partial}{\partial Z}\right)^2 U - \lambda K'$$

or of $$(U_{11}, U_{22}, U_{33}, U_{23}, U_{31}, U_{12})(X', Y', Z')^2 - \lambda K',$$

we have $$4\lambda^3 - I(U)\lambda + J(U),$$

where $$I(U) = U_{11}U_{33} - 4U_{12}U_{23} + 3U_{22}^2,$$

$$J(U) = \begin{vmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{vmatrix}.$$

Expanding $I(U)$ in the form $(a_{11}, a_{22}, a_{33}, a_{23}, a_{31}, a_{12})(X, Y, Z)^2$, we find

$$a_{11} = a_0a_4 - 4a_1a_3 + 3a_2^2, \quad 2a_{23} = a_1a_6 - 3a_2a_5 + 2a_3a_4,$$
$$a_{22} = a_1a_5 - 4a_2a_4 + 3a_3^2, \quad 2a_{31} = a_0a_6 - 4a_1a_5 + 7a_2a_4 - 4a_3^2,$$
$$a_{33} = a_2a_6 - 4a_3a_5 + 3a_4^2, \quad 2a_{12} = a_0a_5 - 3a_1a_4 + 2a_2a_3;$$

also

$$J(U) = \begin{vmatrix} a_0x + a_1y + a_2z, & a_1x + a_2y + a_3z, & a_2x + a_3y + a_4z \\ a_1x + a_2y + a_3z, & a_2x + a_3y + a_4z, & a_3x + a_4y + a_5z \\ a_2x + a_3y + a_4z, & a_3x + a_4y + a_5z, & a_4x + a_5y + a_6z \end{vmatrix}.$$

Operating with Π on $I(U)$, we get

$$I_2 = a_0a_6 - 6a_1a_5 + 15a_2a_4 - 10a_3^2,$$

viz., the quadratic invariant of the sextic; and therefore

$$\Pi(I(U) + \tfrac{1}{6}I_2K) \equiv 0.$$

Also $\Pi J(U) = L_X$ becomes L_x on transformation.

Again, if we form the discriminant of

$$I(U) + \tfrac{1}{6} I_2 K - \lambda K,$$

we have
$$4\lambda^3 - I_4 \lambda + I_6,$$

where I_4 and I_6, the invariants of I_x, are invariants of the fourth and sixth orders of the sextic, the general form of all such invariants being

$$lI_4 + mI_2^2, \quad lI_2 I_4 + mI_2^3 + nI_6.$$

The invariants which Salmon (*Higher Algebra*, p. 262) selects as fundamental are the invariants $-S$ and T of the cubic curve U (*Higher Plane Curves*, Arts. 220, 221; 3rd ed.).

The condition that the cubic and conic should touch is expressed by the vanishing of an invariant I_{10}, and this invariant is the discriminant of the sextic.

The condition that three connectors of the six points of intersection of U and K should meet in a point is expressed by the vanishing of an invariant I_{15}; this is the skew invariant of the sextic, and may be obtained as the invariant R_{123} of Art. 217 for the combined system

$$I(U) + \tfrac{1}{6} I_2 K, \quad K, \quad \Pi J(U).$$

The covariant I_x may also be obtained from the curve $U_1 U_3 - U_2^2$, which transforms into H_x; for, reducing by the relation $U_{31} = U_{22}$, we find

$$\tfrac{1}{4}\Pi(U_1 U_3 - U_2^2) = U_{11} U_{33} - 4U_{12} U_{23} + 3U^2_{22} = I(U).$$

The covariant L_X may also be obtained by substituting D_Z, $-2D_Y$, D_X for X, Y, Z in $I(U)$, and operating on U.

219. **Geometrical Representation of the Jacobian.**—In this Article we propose to find a curve which intersects the fixed conic K in points which represent the roots of the Jacobian of two quantics ϕ and ψ of the n^{th} degree.

Resolving $\frac{\psi}{\phi}$ into partial fractions, and then differentiating, we have

$$J(\phi, \psi) = n\phi^2 \sum_{r=1}^{r=n} \frac{\psi(\alpha_r)}{\phi'(\alpha_r)} \frac{1}{(x-\alpha_r y)^2}.$$

Now, passing to the ternary variables, $J(\phi, \psi)$ transformed becomes

$$J = n(T_1 T_2 T_3 \ . \ . \ . \ T_n) \sum_{r=1}^{r=n} \frac{\psi(\alpha_r)}{\phi'(\alpha_r)} \frac{1}{T_r},$$

where $\qquad T_r = X - \alpha_r Y + \alpha_r^2 Z$, and $\phi(\alpha_r) = 0$.

The curve J plainly passes through all the intersections of the tangents to K at the points $\phi = 0$. Moreover, interchanging ϕ and ψ, this curve passes through all the intersections of the tangents to K at the points $\psi = 0$, J only changing its sign by this interchange. This curve, therefore, passing through the $n(n-1)$ vertices of the two circumscribing polygons, intersects the conic K in $2(n-1)$ points determined by $J(\phi, \psi) = 0$.

It is important to notice that the equation of the curve J does not alter when $\lambda\phi + \psi$ is substituted for ψ, proving that there are an infinite number of polygons circumscribing K and inscribed in J, the points of contact of their sides being determined by the equation $\lambda\phi + \psi = 0$, where λ may have any value; also the curve J of the $n - 1^{st}$ degree is completely fixed by the $2(n-1)$ Jacobian points and the $\frac{n(n-1)}{2}$ vertices of one circumscribing polygon, since it is determined by $\frac{(n-1)(n+2)}{2}$ arbitrary points.

EXAMPLES.

1. If a quartic u have a double factor, prove geometrically that this factor is a double factor of H_x, and show that two of the quadratic factors of G_x have real roots when the roots of u are all real or all imaginary, also that only one factor has real roots when two roots of u are real and two imaginary.

2. If a quartic have a square factor, prove geometrically that this factor is a quintuple factor of the covariant G_x; and construct the point on the conic K which corresponds to the remaining root of the equation $G_x = 0$.

3. Prove that the quadratic factors of the sextic covariant of the quartic $\phi(x)$ expressed in terms of the roots may be written in the form

$$\frac{(x-\alpha_1 y)^2}{\phi'(\alpha_1)} + \frac{(x-\alpha_2 y)^2}{\phi'(\alpha_2)}, \ \&c.$$

Let
$$T_r = X - \alpha_r Y + \alpha_r^2 Z;$$
we have then
$$\frac{T_1}{\phi'(\alpha_1)} + \frac{T_2}{\phi'(\alpha_2)} + \frac{T_3}{\phi'(\alpha_3)} + \frac{T_4}{\phi'(\alpha_4)} \equiv 0 \text{ (by Euler's theorem)};$$

(Ex. 4, p. 173, Vol. I.)

but the sides of the self-conjugate triangle corresponding to the points α_1, α_2, α_3, α_4 on K are the diagonals of the quadrilateral formed by the tangents T_1, T_2, T_3, T_4, and the equation of one of the sides is therefore

$$\frac{T_1}{\phi'(\alpha_1)} + \frac{T_2}{\phi'(\alpha_2)} = 0, \quad \text{or} \quad \frac{T_3}{\phi'(\alpha_3)} + \frac{T_4}{\phi'(\alpha_4)} = 0.$$

Returning now to the binary system of variables, we have the required form.

4. Resolve the quartic as in Art. 186 by finding the tangents to the conic K where U meets it, U and K having been expressed as sums of squares.

5. Determine the condition that $\lambda u + \mu v$ should have two square factors, where u and v are binary quartics.

Transforming to ternary variables, we have in this case

$$\lambda U + \mu V + \nu K \equiv (\alpha X + \beta Y + \gamma Z)^2;$$

consequently, every term in the tangential form of $\lambda U + \mu V + \nu K$ must vanish, giving six equations to eliminate λ^2, μ^2, ν^2, $\mu\nu$, $\nu\lambda$, $\lambda\mu$; hence the required condition is determined.

6. If u, v, and w be three binary quartics, prove that four quartics can be found such that

$$\lambda u + \mu v + \nu w \equiv (\alpha x^2 + 2\beta xy + \gamma y)^2.$$

Passing to the ternary variables, we have to prove that four lines can be found such that

$$\lambda U + \mu V + \nu W + \rho K \equiv (\alpha X + \beta Y + \gamma Z)^2.$$

These lines are the common tangents to two known conics (see Salmon's *Conics*, Ex. 3, Art. 373).

7. Apply the geometrical transformation of Art. 212 to prove that the Tschirnhausen transformation, $z = (\alpha x^2 + 2\beta x + \gamma)/(\alpha' x^2 + 2\beta'\gamma + \gamma')$, transforms a quadric into one having the same absolute invariant as a quartic whose roots are κ, ρ_1, ρ_2, ρ_3, or in fact to prove the theorem of Art. 198.

Make the numerator and denominator in the expression for z homogeneous in x, y; replace z by $-\lambda$, and transform: the Tschirnhausen transformation becomes

$$L + \lambda L' = 0,$$

where
$$L = \alpha X + \beta Y + \gamma Z, \quad L' = \alpha' X + \beta' Y + \gamma' Z.$$

If X, Y, Z be eliminated from the equations $L + \lambda L' = 0$, $U = 0$, $K = 0$, we shall have the transformed quartic in λ, which, considered geometrically, determines the lines drawn from the point of intersection P of L and L' to the points of intersection A, B, C, D of U and K. Again, if κ be so determined that the conic $U + \kappa K$ pass through the point P, the anharmonic ratio of the lines PA, PB, PC, PD is equal to the anharmonic ratio of the lines TD, DA, DB, DC, where TD is the tangent to $U + \kappa K$ at D; that is, of the lines

$$t + \kappa t', \quad t + \rho_1 t', \quad t + \rho_2 t', \quad t + \rho_3 t',$$

where t and t' are the tangents to U and K at D. As the anharmonic ratio is the same, the absolute invariant is the same for both quartics, viz. the given quartic and the quartic whose roots are κ, ρ_1, ρ_2, ρ_3.

8. Transform a quartic into one having three roots in common with its reducing cubic.

This transformation is suggested by the last example and may be effected by putting $L = t$, $L' = t'$, where t, t' are the tangents to U and K at the point of intersection D corresponding to δ. As $t + \rho_1 t'$, $t + \rho_2 t'$, $t + \rho_3 t'$ are the lines joining the point corresponding to δ to the points corresponding to α, β, γ respectively, transforming to the binary system, we take

$$\frac{\xi}{\eta} = -\frac{t}{t'} = -\frac{1}{24}\,\frac{\left(x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}\right)^2 U_x}{(xy' - x'y)^2} = -\frac{1}{24}\,\frac{\left(\delta\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 U_x}{(x - \delta y)^2}$$

$$= -\frac{1}{2}\cdot\frac{(a\delta^2 + 2b\delta + c)\,x^2 + 2xy\,(b\delta^2 + 2c\delta + d) + y^2\,(c\delta^2 + 2b\delta + e)}{(x - \delta y)^2}$$

which is satisfied by putting $\xi/\eta = \rho_1$ or ρ_2 or ρ_3 and $x/y = \alpha$ or β or γ, respectively. Dividing both numerator and denominator by $x - \delta y$, we take $\eta = x - \delta y$ and

$$\xi = -\tfrac{1}{2}(a\delta^2 + 2b\delta + c)\,x - \tfrac{1}{2}\{(a\alpha^2 + 2b\alpha + c)\,\alpha + 2\,(b\alpha^2 + 2c\alpha + d)\}y$$
$$\equiv -\tfrac{1}{12}a\Sigma(\delta - \alpha)(\delta - \beta)\,.\,x + \tfrac{1}{12}a\Sigma\alpha(\delta - \beta)(\delta - \gamma)\,.\,y,$$

where the summation is with respect to α, β, γ.

This transforms U_x to $k\eta\,(4\xi^3 - I\xi\eta^2 + \eta^3)$, where

$$1/k = -\tfrac{a^2}{16}(\delta - \alpha)^2(\delta - \beta)^2(\delta - \gamma)^2.$$ (See Ex. 32, page 213.)

9. Let three points a, b, c be taken on the conic K given by the equations

$$\rho x = \phi^2, \quad \rho y = 2\phi, \quad \rho z = 1,$$

the values of ϕ at these points being α, β, γ, the roots of a cubic U; prove the following constructions for determining the points on the conic corresponding to the roots of the cubic covariant G_x and the Hessian H_x:—

1°. Let tangents be drawn to the conic K at the points a, b, c, forming a triangle ABC; the lines Aa, Bb, Cc meet the conic at points a', b', c', corresponding to the roots of G_x.

2°. If tangents are drawn to K at a', b', c', forming a triangle $A'B'C'$, the four triangles abc, $a'b'c'$, ABC, $A'B'C'$ are homologous, and their axis of homology meets the conic K at the points corresponding to the roots of H_x.

3°. From the foregoing constructions, prove that U_x and G_x have the same Hessian H_x, and that the roots of H_x are imaginary when the roots of U_x are real. —*Dublin Exam. Papers, Bishop Law's Prize*, 1879.

Let the tangents at the points α, β, γ on the fixed conic K be T_α, T_β, T_γ; we have then

$$\rho T_\alpha = (\phi - \alpha)^2, \quad \rho T_\beta = (\phi - \beta)^2, \quad \rho T_\gamma = (\phi - \gamma)^2;$$

whence, eliminating ϕ, the equation of K is

$$(\beta - \gamma)\sqrt{T_\alpha} + (\gamma - \alpha)\sqrt{T_\beta} + (\alpha - \beta)\sqrt{T_\gamma} = 0.$$

Now the equations of the lines Aa, Bb, Cc are

$$(\gamma - \alpha)^2 T_\beta - (\alpha - \beta)^2 T_\gamma = 0, \text{ \&c., \&c.,}$$

and the points where Aa meets the conic K are given by the equation

$$(\gamma - \alpha)^2 (\phi - \beta)^2 = (\alpha - \beta)^2 (\phi - \gamma)^2.$$

Solving $\quad \phi = \alpha$ and $(\beta + \gamma - 2\alpha)\phi = 2\beta\gamma - \gamma\alpha - \alpha\beta$,

the second value of ϕ being the root of G_x corresponding harmonically to α.

Again, the polar of the point of intersection of Aa, Bb, Cc is the axis of homology in 2°, and its equation is

$$(\beta - \gamma)^2 T_\alpha + (\gamma - \alpha)^2 T_\beta + (\alpha - \beta)^2 T_\gamma = 0,$$

which line meets K at the points determined by the equation

$$(\beta - \gamma)^2 (\phi - \alpha)^2 + (\gamma - \alpha)^2 (\phi - \beta)^2 + (\alpha - \beta)^2 (\phi - \gamma)^2 = 0,$$

and this is the equation of the Hessian of U.

10. Resolve the skew invariant of the quadratic and cubic into three factors, in terms of the roots, and give its geometrical interpretation.

The skew invariant is expressed in the form

$$V_D{}^3 (U_x G_x). \qquad \text{(Art. 191.)}$$

Now combining the factors of U_x and G_x which correspond harmonically, $U_x G_x$ can be expressed as the product of three quadratics l, m, n, where

$$l = (\beta + \gamma - 2\alpha) x^2 - 2(\beta\gamma - \alpha^2) xy + \alpha(2\beta\gamma - \gamma\alpha - \alpha\beta) y^2,$$

with similar values for m and n.

Now we can prove $V_D{}^3 (lmn) = k V_D l \,.\, V_D m \,.\, V_D n$, where k is a numerical multiplier.

This is more easily seen by employing the ternary system of variables; in that case, if

$$V_x = x^2 - (\mu + \nu) xy + \mu\nu y^2,$$

$6V_D$, considered as an operator on LMN transforms to

$$\mu\nu D_X + (\mu + \nu) D_Y + D_Z \equiv \Upsilon,$$

because, as we shall see, $\Pi LMN \equiv 0$, and hence

$$\Upsilon^3 (LMN) = 6\Upsilon L \,.\, \Upsilon M \,.\, \Upsilon N,$$

L being l transformed, &c.

Now,

$$\Upsilon(L) = \begin{vmatrix} a^2 & 2a & 1 \\ \mu\nu & \mu + \nu & 1 \\ \beta\gamma & \beta + \gamma & 1 \end{vmatrix},$$

which vanishes when a determines a focus of the involution of the points μ, ν and β, γ; or again when V_x and l determine four harmonic points on a line; or again when one of the lines Aa, Bb, Cc, in question 9, and the line corresponding to V_x, are conjugate with regard to the fixed conic K; the skew invariant also vanishing in these cases.

That
$$\Pi(LMN) \equiv 0$$

is now easily seen by transforming as in question 9 to the variables

$$X' = (\beta - \gamma)^2 T_a = (\beta - \gamma)^2 (X - aY + a^2 Z), \text{ \&c.},$$

then L transforms to $(Y' - Z') / (\beta - \gamma)$, and Π to

$$(\beta - \gamma)^2 (\gamma - a)^2 (a - \beta)^2 \left(\frac{\partial^2}{\partial Y' \partial Z'} + \frac{\partial^2}{\partial Z' \partial X'} + \frac{\partial^2}{\partial X' \partial Y'} \right);$$

whence $\Pi(LMN) \equiv 0$.

11. Two triads of points are taken on the conic K determined by the roots of two cubics U and V, if triangles are formed by the tangents to K at these points, prove that the conic circumscribing these two triangles touches the conic K when the combinant Q of the two cubics vanishes, and that their combinant P vanishes when the circumscribing conic meets the conic K in four equianharmonic points.

12. Determine the condition that any two quadratic factors, viz.,

$$(x - ay)(x - \beta y), \quad (x - \gamma y)(x - \delta y)$$

of a quartic u should form with a given quadratic $\lambda x^2 + 2\mu xy + \nu y^2$ a system in involution.

Transforming, the three corresponding lines must meet in a point, which point is one of the vertices of the common self-conjugate triangle of the conics U and K. The tangential equation of these points is $J(\Sigma, \Sigma', \Phi) = 0$, which is therefore the required condition, the tangential form of $\rho U + K$ being $\rho^2 \Sigma + \rho \Phi + \Sigma'$ (Art. 217).

This condition may also be put under the form

$$\left(\lambda \frac{\partial^2}{\partial y^2} - 2\mu \frac{\partial^2}{\partial x \partial y} + \nu \frac{\partial^2}{\partial x^2} \right)^3 G_x = 0,$$

as we proceed to show.

If
$$\Upsilon = \lambda \frac{\partial^2}{\partial y^2} - 2\mu \frac{\partial^2}{\partial x \partial y} + \nu \frac{\partial^2}{\partial x^2},$$

and $G_x \equiv lmn$ when resolved into its quadratic factors, we have

$$\Upsilon^3 G_x = 6\Upsilon l \,.\, \Upsilon m \,.\, \Upsilon n,$$

for transforming to the ternary variables,

$$\Upsilon = 6\left(\lambda \frac{\partial}{\partial Z} - 2\mu \frac{\partial}{\partial Y} + \nu \frac{\partial}{\partial X}\right)$$

when applied to a function $\phi(X, Y, Z)$ such that $\Pi\phi \equiv 0$. Now l, m, n become three lines, L, M, N, which form a self-conjugate triangle with reference to K, and $\Pi LMN \equiv 0$ in the case of any three lines which are mutually conjugate; whence

$$\Upsilon^3 LMN \text{ reduces to } 6\Upsilon L \,.\, \Upsilon M \,.\, \Upsilon N,$$

and $\Upsilon L = 0$ is the condition that the lines $\lambda X + \mu Y + \nu Z$ and L should be conjugate with respect to K, or that $\lambda X + \mu Y + \nu Z$ should pass through the pole of L, or the condition that two quadratic factors of u should form with $\lambda x^2 + 2\mu xy + \nu y^2$ a system in involution.

13. Prove that the quartics

$$v \equiv (\alpha_1 x^2 + 2\beta_1 xy + \gamma_1 y^2)(\alpha_3 x^2 + 2\beta_3 xy + \gamma_3 y^2) - (\alpha_2 x^2 + 2\beta_2 xy + \gamma_2 y^2)^2, \quad (1)$$

$$u \equiv (\alpha_1 x^2 + 2\alpha_2 xy + \alpha_3 y^2)(\gamma_1 x^2 + 2\gamma_2 xy + \gamma_3 y^2) - (\beta_1 x^2 + 2\beta_2 xy + \beta_3 y^2)^2 \quad (2)$$

have the same invariants.

Transforming (2) to the ternary system, we have the conic

$$(\alpha_1 X + \alpha_2 Y + \alpha_3 Z)(\gamma_1 X + \gamma_2 Y + \gamma_3 Z) - (\beta_1 X + \beta_2 Y + \beta_3 Z)^2 + \kappa(4ZX - Y^2),$$

where $3\kappa = I_{22} - I_{13}$ has been found so that $\Pi U \equiv 0$, and κ is the same for U and V.

For shortness, we write

$$L \equiv \alpha_1 X + \alpha_2 Y + \alpha_3 Z,\ M \equiv \beta_1 X + \beta_2 Y + \beta_3 Z,\ N \equiv \gamma_1 X + \gamma_2 Y + \gamma_3 Z. \quad (3)$$

Forming the discriminant of

$$U + \lambda(4ZX - Y^2) \equiv LN - M^2 + (\kappa + \lambda)(4ZX - Y^2),$$

$$\left.\begin{aligned} N\alpha_1 - 2M\beta_1 + L\gamma_1 + 4(\lambda + \kappa) Z &= 0 \\ N\alpha_2 - 2M\beta_2 + L\gamma_2 - 2(\lambda + \kappa) Y &= 0 \\ N\alpha_3 - 2M\beta_3 + L\gamma_3 + 4(\lambda + \kappa) X &= 0 \end{aligned}\right\} \quad (4)$$

and eliminating X, Y, Z; or eliminating the six quantities X, Y, Z, L, M, N by means of the three additional equations (3), and putting $\lambda + \kappa = \lambda'$, the resultant is obtained in the form

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 & 0 & 0 & +4\lambda' \\ \alpha_2 & \beta_2 & \gamma_2 & 0 & -2\lambda' & 0 \\ \alpha_3 & \beta_3 & \gamma_3 & +4\lambda' & 0 & 0 \\ 0 & 0 & -1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & +\frac{1}{2} & 0 & \beta_1 & \beta_2 & \beta_3 \\ -1 & 0 & 0 & \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} \equiv \Delta(\lambda').$$

If we had operated similarly on the quartic (1), we should have obtained the same resultant $\Delta(\lambda')$, the form the determinant takes in this case being obtained by dividing the first three rows of $\Delta(\lambda')$ by $-4\lambda'$, multiplying the first three columns by $-4\lambda'$, moving the last three columns to the front, and then the last three rows to the top. Whence it follows that the invariants are the same in both cases.

To expand $\Delta(\lambda')$ we replace L, M, N by their values in equations (4), and then eliminate X, Y, Z, thus obtaining

$$\begin{vmatrix} I_{11} & I_{12} & I_{13}+2\lambda' \\ I_{12} & I_{22}-\lambda' & I_{23} \\ I_{13}+2\lambda' & I_{23} & I_{33} \end{vmatrix}, \text{ where } 2I_{pq} = a_p\gamma_q + a_q\gamma_p - 2\beta_p\beta_q.$$

This determinant becomes, when expanded,

$$4\lambda'^3 - 4(I_{22}-I_{13})\lambda'^2 - \{I_{11}I_{33} - I^2_{13} + 4(I_{13}I_{22} - I_{12}I_{23})\}\lambda' + \begin{vmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{vmatrix}.$$

Note when $\lambda + \kappa$ is substituted for λ', this equation has no second term, and every coefficient must be the same for both quartics, as may be verified directly. (See Zeuthen's solution, *Proceedings of the London Mathematical Society*, vol. x., p. 196.)

14. Determine the condition that three quadratics in terms of their invariants should by linear transformation be simultaneously reducible to the forms

$$\frac{\partial^2\phi}{\partial x^2}, \quad \frac{\partial^2\phi}{\partial x\partial y}, \quad \frac{\partial^2\phi}{\partial y^2},$$

Ans. $I_{11}I_{33} - 4I_{12}I_{23} + I_{22}{}^2 + 2I_{22}I_{31} = 0.$

$$2I_{pq} = a_p\gamma_q + a_q\gamma_p - 2\beta_p\beta_q.$$

15. Prove that the condition in Ex. 14 is the same for the following two sets of quadratics :—

$$a_1x^2 + 2\beta_1xy + \gamma_1y^2, \quad a_2x^2 + 2\beta_2xy + \gamma_2y^2, \quad a_3x^2 + 2\beta_3xy + \gamma_3y^2,$$

and

$$a_1x^2 + 2a_2xy + a_3y^2, \quad \beta_1x^2 + 2\beta_2xy + \beta_3y^2, \quad \gamma_1x^2 + 2\gamma_2xy + \gamma_3y^2.$$

The condition in Ex. 14 can be put under the form

$$(I_{22} - I_{13})^2 + I_{11}I_{33} - I_{13}{}^2 + 4(I_{13}I_{22} - I_{12}I_{23}),$$

which is at once expressible by the coefficients of $\Delta(\lambda')$ in Ex. 13.

The geometrical interpretation of this condition is that a triangle can be inscribed in the harmonic conic of U and K and circumscribed to K.

For, replacing L, M, N in Ex. 13 by U_1, U_2, U_3, as supposed in Ex. 14, where

$$U \equiv (a, c, e, d, c, b)(X, Y, Z)^2,$$

$$U_1U_3 - U_2{}^2 \equiv (I_{11}, I_{22}, I_{33}, I_{23}, I_{31}, I_{12})(X, Y, Z)^2$$

becomes the harmonic conic F of U and K (Art. 214); also if the discriminant of $\lambda K + F$ or $\Delta(\lambda)$ be written in the usual form

$$\Delta\lambda^3 + \Theta\lambda^2 + \Theta_1\lambda + \Delta_1,$$

the condition in Ex. 14 may be written thus—

$$\Theta^2 - 4\Delta\Theta_1 = 0,$$

which is the well-known invariant condition that a triangle can be inscribed in one conic and circumscribed to another. (See Salmon's *Conic Sections*, Art. 376.)

16. If U and V be the ternary forms of two biquadratics u and v, and F their harmonic conic, prove that $\Pi(F) = 0$ is the condition that u and v should be two first emanants of a binary quintic.

CHAPTER XX.

THEORY OF SUBSTITUTIONS AND GROUPS.

SECTION I.—SUBSTITUTIONS IN GENERAL.

220. **Definitions—Notation.**—If n symbols $x_1, x_2, x_3, \ldots x_n$ be given, and if each symbol be replaced by some one or other from the same set, so that the result is a new arrangement of the same n symbols, the operation of passing from the first to the second arrangement is called a *substitution*. The symbols $x_1, x_2, \ldots x_n$ are to be regarded as entirely independent quantities, and are referred to as the *variables*, or the *elements* affected by the substitution.

If the operation be denoted by S, a substitution S can be represented as follows:—

$$S \equiv \begin{pmatrix} x_1 & x_2 & x_3 & \ldots & x_n \\ x_\alpha & x_\beta & x_\gamma & \ldots & x_\lambda \end{pmatrix},$$

where the two horizontal lines contain the same set of n letters, and the operation consists in replacing any letter in the upper line by that which stands under it in the lower line. The operation may be supposed to be applied to a function $\phi(x_1, x_2, \ldots x_n)$ of the variables, in which case the resulting function $S\phi$ will be obtained by replacing x_1 by x_α wherever it occurs in ϕ, x_2 by x_β, and so on. In the case of any letter which is not displaced by the substitution under consideration, the two symbols in the same vertical line will be identical. Since the suffixes of x admit of only $1 \cdot 2 \cdot 3, \ldots n \equiv N$ permutations, this is the total possible number of distinct substitutions. In this number is included that arrangement in which the order of the suffixes is the same in both horizontal lines, viz., that in which no letter is displaced by the operation. Such a substitution, which affects no element, is called the *identical substitution*, or the *substitution unity*, and may be denoted by $S \equiv 1$.

It will usually be found convenient in practice to denote the symbols operated on by single letters $a, b, c, \ldots$, or by the numbers 1, 2, 3, . . . simply, the symbol x being omitted.

221. **Circular Substitutions.**—The notation above explained admits of simplification. Consider, for example, the substitution

$$S \equiv \begin{pmatrix} a\,b\,c\,d\,e\,f \\ b\,c\,d\,e\,f\,a \end{pmatrix},$$

in which each symbol is replaced by that which follows it in the first line, the last letter f being replaced by the first. Such is called a *circular substitution*, and is denoted simply by the letters of the first line enclosed in a bracket, thus—

$$S \equiv (a\,b\,c\,d\,e\,f).$$

It is clear that S can be written in several different ways, and that any of the letters involved may stand first, provided the cyclical order be preserved: thus

$$S \equiv (bcdefa) \equiv (cdefab) \equiv (defabc) \equiv (efabcd) \equiv (fabcde).$$

Now it is easy to see that *every substitution can be resolved into one or more circular substitutions*. For, in effecting any substitution S, if a letter a in the upper line be found replaced by b, and b in its turn by c, and so on, in continuing this process, we come necessarily to a letter (h, say) which is found replaced by a. The result of the operation so far is the circular substitution $(abc \ldots h)$. If the letters be not all exhausted by this process, we select a letter from those which remain, and form in a similar manner a new circular substitution; and so on, as long as any new symbols remain.

If we denote by $C_1, C_2, \ldots C_s$ the different substitutions obtained in this way, we may write

$$S \equiv C_1 C_2 C_3 \ldots C_s,$$

and S may be said to be resolved into its circular factors. These factors are called the *cycles* of S. Cycles which contain two letters only are called *transpositions*. As an example, we take the substitution

$$S \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 6 & 1 & 2 & 7 & 5 & 4 & 3 \end{pmatrix}.$$

Starting with the symbol 1 in the upper line we obtain immediately the cycle (183), and proceeding in a similar manner with 2 we obtain the cycle (26574); hence

$$S \equiv (183)\,(26574).$$

It is clear that the order in which the operations are conducted is indifferent, since no cycle affects any of the elements contained in any other, and therefore the order in which the factors of S are written is indifferent.

If all the elements are involved in the first operation alone, the substitution is itself circular, e.g.,

$$S \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 6 & 7 & 5 & 2 & 1 & 4 \end{pmatrix} \equiv (1374526).$$

If the position of any element is unaltered by a substitution, this element may be enclosed in brackets by itself when the substitution is expressed as a product of cycles, or it may be omitted altogether, e.g.,

$$S \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 1 & 5 & 2 \end{pmatrix} \equiv (134)\,(26)\,(5) \equiv (134)\,(26).$$

Here (5) being the identical substitution $\equiv 1$ may be replaced by unity. Although an element constituting a cycle by itself can be replaced by unity, it is often necessary to retain it in order to show that this element was amongst those which were subject to the operation.

A circular substitution S can be repeated any number of times on the same elements, and the successive operations denoted by S^2, S^3, &c. We have, for example,

$$S \equiv \begin{pmatrix} a & b & c & d & e & f \\ b & c & d & e & f & a \end{pmatrix},\; S^2 \equiv \begin{pmatrix} a & b & c & d & e & f \\ c & d & e & f & a & b \end{pmatrix},\; S^3 \equiv \begin{pmatrix} a & b & c & d & e & f \\ d & e & f & a & b & c \end{pmatrix},$$

Proceeding, we find $S^6 = 1$. If, in general, ρ is the lowest integer such that $S^\rho = 1$, the substitution S is said to be of the order ρ; it is clear, therefore, that the order of a circular substitution is equal to the number of elements it displaces.

For two elements α, β we have $(\alpha\beta) = (\beta\alpha)$, and $(\alpha\beta)^2 = 1$.

For three elements α, β, γ we have $(\alpha\beta\gamma)^2 = (\alpha\gamma\beta)$; $(\alpha\beta\gamma)^3 = 1$.

222. **Products and Powers of Substitutions.**—If two or more substitutions $S_1, S_2, \ldots S_j$ be operated in succession on a given set of elements, the result is a new arrangement which might have been arrived at by one single substitution S. This substitution may be called the *product* of the former set, and we may write $S = S_1S_2, \ldots S_j$, the component factors being applied in the order $S_1, S_2, \ldots$, viz., FROM LEFT TO RIGHT. When a substitution is resolved into its component *cycles*, as in the preceding Article, we saw that the order of the factors is indifferent, no element being common to any two of the cycles. But, in general, in a product of substitutions where the same element may occur in two or more of the factors $S_1, S_2, \ldots$, it is most important to observe that the commutative law of algebraic multiplication does not hold good, and that the order of the factors must be preserved. With three elements, for example, the student will easily verify that the product (12) (13) is a different substitution from the product (13) (12). While the commutative law of algebra fails, the associative law holds good, viz., $S_1S_2 . S_3 = S_1 . S_2S_3$; for if S_1 changes any element a into b, and S_2 changes b into c, which again is changed into d by means of S_3, the substitution of d for a is the final result whether this be supposed effected by first changing a into c (by means of S_1S_2), and then c into d, or first changing a into b, and then b into d (by means of S_2S_3).

The result of operating the same substitution S any number of times, p, in succession may be represented by S^p; and we have clearly the equation $S^pS^q = S^{p+q} = S^qS^p$. The *inverse* of a given substitution S is one which reverses the order of procedure in S, and is denoted by the symbol S^{-1}. Thus, if

$$S = \begin{pmatrix} a_1 & a_2 & a_3 & . & . & a_n \\ b_1 & b_2 & b_3 & . & . & b_n \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} b_1 & b_2 & b_3 & . & . & b_n \\ a_1 & a_2 & a_3 & . & . & a_n \end{pmatrix},$$

we have clearly $$SS^{-1} = S^{-1}S = 1.$$

Since the total number of possible substitutions is limited, some repetition of S must reproduce the original arrangement of the elements. If ρ is the lowest integer such that $S^\rho = 1$, S is said to be of the *order* ρ, and the series of substitutions is limited as follows :—

$$1, S, S^2, S^3, \ . \ . \ . \ S^{\rho-1}.$$

The extension of this mode of expression to negative exponents may be obtained by writing S^{-p} in the form $S^{k\rho-p}$, where ρ is the order of S, and consequently $S^{k\rho} = 1$. We have, then, $S^pS^{-p} = S^pS^{k\rho-p} = S^{k\rho} = 1$, and the substitutions S^p and S^{-p} cancel one another.

Any circular substitution can be represented as a product of transpositions, for it is clear that the operation $(abcdef)$ can be conducted by first interchanging a and b, then interchanging a and c, then a and d, and so on. We may write, therefore,

$$(abcdef) = (ab)\,(ac)\,(ad)\,(ae)\,(af),$$

from which it appears that any cycle can be resolved into a product of transpositions, in number one less than the number of elements contained in the cycle. The order of the factors in any such product is important, these factors not being commutable amongst one another. It follows immediately that *every substitution can be expressed as a product of transpositions*, for each of its cycles can be so expressed.

If a substitution S affecting n elements contains k cycles, it can be easily inferred that S can be expressed as a product of $n - k$ transpositions. It should be observed, however, that the same substitution can be expressed in a great variety of ways as a product of transpositions. It will appear in the sequel that, however variously expressed, the number of transpositions in any given substitution preserves the same *parity* ; that is to say, if once even, it is always even ; if once odd, always odd.

Examples.

1. Resolve into its cycles

$$S \equiv \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{11} & a_7 & a_5 & a_{12} & a_1 & a_2 & a_3 & a_4 & a_6 & a_9 & a_{10} & a_8 & a_{14} & a_{13} & a_{15} \end{pmatrix}.$$

Ans. $S \equiv (a_1\, a_{11}\, a_{10}\, a_9\, a_8\, a_2\, a_7\, a_3\, a_5)\,(a_4\, a_{12}\, a_6)\,(a_{13}\, a_{14})\,(a_{15})$.

The appearance of the factor $(a_{15}) \equiv 1$ in the result shows that this element was amongst those subject to the operation.

2. Express as a product of transpositions

$$S \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 \\ 3 & 8 & 6 & 9 & 2 & 4 & 0 & 5 & 1 & 7 \end{pmatrix}.$$

Ans. $S \equiv (13)\,(16)\,(14)\,(19)\,(28)\,(25)\,(70)$.

3. If a circular substitution C be multiplied by a transposition T, one of whose elements is contained in C and the other not, the resulting substitution CT is circular.

Taking a_1 as the common element, we may write

$$C \equiv (a_1 a_2 a_3 \ . \ . \ . \ a_i), \quad T \equiv (a_1 a_j).$$

The effect of C is to replace the arrangement $a_1, a_2, \ . \ . \ . \ a_i, a_j$ by $a_2, a_3, \ . \ . \ . \ a_i, a_1, a_j$, and of T to interchange a_1 and a_j in the latter. We have then

$$CT \equiv \begin{pmatrix} a_1 & a_2 & . \ . & a_{i-1} & a_i & a_j \\ a_2 & a_3 & . \ . & a_i & a_j & a_1 \end{pmatrix} \equiv (a_1\, a_2 \ . \ . \ . \ a_i a_j).$$

4. If a circular substitution C be multiplied by a transposition T, both of whose elements are contained in C, the resulting substitution CT is the product of two cycles having no common element.

We may take

$$C \equiv (a_1\, a_2 \ . \ . \ . \ a_i\, b_1\, b_2 \ . \ . \ . \ b_j), \quad T \equiv (a_1\, b_1).$$

Proceeding as in the previous example, we readily find

$$CT \equiv (a_1\, a_2 \ . \ . \ . \ a_i)\,(b_1\, b_2 \ . \ . \ . \ b_j).$$

5. If a substitution S be multiplied by a transposition T, whose elements are contained one in each of two different cycles, C, C', of S, the product $CC'T$ is one unbroken cycle of all the elements in C and C'.

This follows at once from the last example by multiplying both sides of the resulting equation into T, since $T^2 = 1$.

6. If any substitution S is the product of r transpositions, and if it be multiplied by a transposition T, the product ST will consist either of $r + 1$ or $r - 1$ transpositions.

If S affects n elements, and contains k cycles, we have, as stated above, $r = n - k$. If T introduces two new elements, we have one additional transposition, hence $r + 1$ in all. There are three cases remaining; according as (1) T introduces one new element only, or (2) two elements already contained in the same cycle of S, or (3) two already contained in different cycles of S. These cases are discussed in the three preceding examples; and it is readily inferred that the number of transpositions in ST is always $r + 1$, except when both elements of T occur in the same cycle of S, in which case n is unaltered, and k becomes $k + 1$; r therefore becomes

$$n - (k + 1) = n - k - 1 = r - 1.$$

It appears from this example, that however S be expressed as a product of transpositions, the effect of multiplication by a single additional transposition is to change its parity, viz., from odd to even, or even to odd.

7. The order of a substitution S is equal to the least common multiple of the orders of its cycles.

Let $$S = C_1C_2C_3 \ . \ . \ . \ C_j,$$

and μ be any common multiple of the orders of $C_1, C_2, \ . \ . \ . \ .$ Since

$$S^\mu = C_1^\mu C_2^\mu \ . \ . \ . \ C_j^\mu, \quad \text{and} \quad C_1^\mu = C_2^\mu = \ . \ . \ C_j^\mu = 1,$$

we have $S^\mu = 1$; and if ρ be the least value of μ, $S^\rho = 1$; whence ρ is the order of S.

Hence we infer that if the cycles $C_1, C_2, C_3, \ . \ . \ .$ are of the same order, this order is also the order of S. Such substitutions are called *regular*, the same number of letters occurring in each cycle.

8. If a circular substitution S contains p letters, and if μ is prime to p, then S^μ is itself circular.

9. If a circular substitution S contains pq letters, then S^p is a regular substitution consisting of p cycles of q letters each. If, for example,

$$S \equiv (123456), \quad S^2 \equiv (135)(246), \quad S^3 \equiv (14)(25)(36).$$

10. Every regular substitution is a power of a circular substitution.

Take S as in example 7, with the factors

$$C_1 \equiv (a_1b_1c_1 \ . \ . \ . \ l_1), \quad C_2 \equiv (a_2b_2c_2 \ . \ . \ . \ l_2) \ . \ . \ . \ C_j \equiv (a_jb_jc_j \ . \ . \ l_j),$$

i.e., such that the same number of letters are involved in each cycle. If now we write down the circular substitution

$$C \equiv (a_1a_2a_3 \ . \ . \ a_jb_1b_2b_3 \ . \ . \ b_j \ . \ . \ . \ . \ l_1l_2l_3 \ . \ . \ l_j),$$

whose first j letters are the initial letters of the j cycles, the next set the second letters of the successive cycles, and so on, it is easily verified that the j^{th} power of C breaks up into the product of the j successive cycles $C_1, C_2, \ . \ . \ . \ C_j$; hence

$$S = C^j.$$

11. Express the regular substitution

$$S \equiv (1 \ \ 3 \ \ 5 \ \ 12)(2 \ \ 7 \ \ 6 \ \ 11)(4 \ \ 8 \ \ 10 \ \ 9)$$

as a power of a circular substitution.

Ans. $S \equiv (1\ 2\ 4\ 3\ 7\ 8\ 5\ 6\ 10\ 12\ 11\ 9)^3$.

12. Every transposition of the elements $x_1, x_2, x_3, \ldots x_n$ can be expressed by transpositions from the following series of $n-1$, viz. :—

$$(x_1x_2), \quad (x_1x_3), \quad (x_1x_4), \ldots (x_1x_{n-1}), \quad (x_1x_n).$$

For it can be easily verified that, if x_α, x_β are any two elements,

$$(x_\alpha x_\beta) = (x_1x_\alpha)(x_1x_\beta)(x_1x_\alpha).$$

13. Every substitution which can be resolved into an even number of transpositions can be expressed by circular substitutions of the third order.

The given substitution is expressible by products either of the type $(\alpha\beta)(\alpha\gamma)$ or $(\alpha\beta)(\gamma\delta)$; we have $(\alpha\beta)(\alpha\gamma) = (\alpha\beta\gamma)$, and $(\alpha\beta)(\gamma\delta) = (\alpha\beta\gamma)(\alpha\delta\gamma)$, since

$$\begin{aligned}(\alpha\beta\gamma)\ (\alpha\delta\gamma) &= (\alpha\beta\gamma)(\gamma\alpha\delta)\\ &= (\alpha\beta)(\alpha\gamma)(\gamma\alpha)(\gamma\delta)\\ &= (\alpha\beta)(\alpha\gamma)^2(\gamma\delta), \quad \text{and} \quad (\alpha\gamma)^2 = 1.\end{aligned}$$

14. Show that any circular substitution of three of the elements $x_1, x_2, \ldots x_n$ can be expressed by means of the $n-2$ circular substitutions

$$(x_1x_2x_3), \quad (x_1x_2x_4), \ldots . (x_1x_2x_{n-1}), \quad (x_1x_2x_n).$$

Retaining for brevity the suffixes only, we proceed to express $(\alpha\beta\gamma)$ in terms of $(\lambda\mu\alpha)$, $(\lambda\mu\beta)$, and $(\lambda\mu\gamma)$.

$$\begin{aligned}(\alpha\beta\gamma) &= (\alpha\beta)(\alpha\gamma)\\ &= (\alpha\beta)(\alpha\lambda)(\alpha\lambda)(\alpha\gamma), \quad \text{since } (\alpha\lambda)^2 = 1,\\ &= (\alpha\beta\lambda)(\alpha\lambda\gamma)\\ &= (\lambda\alpha\beta)(\lambda\gamma\alpha).\end{aligned}$$

Now making use of this equation to bring a new element μ in a similar manner into each of the cycles on the right-hand side, we have

$$\begin{aligned}(\alpha\beta\gamma) &= (\mu\lambda\alpha)(\mu\beta\lambda)(\mu\lambda\gamma)(\mu\alpha\lambda)\\ &= (\lambda\mu\alpha)^2(\lambda\mu\beta)(\lambda\mu\gamma)^2(\lambda\mu\alpha), \text{ the required expression.}\end{aligned}$$

The following mode of expression can also be easily verified :—

$$(\alpha\beta\gamma) = (\lambda\mu\alpha)(\lambda\mu\gamma)(\lambda\mu\beta)(\lambda\mu\alpha)(\lambda\mu\gamma).$$

223. Similar Substitutions.—Two substitutions which contain the same number of cycles, and the same number of elements in corresponding cycles, are said to be *similar.*

Two substitutions S, T are said to be *commutative* when $ST = TS$.

The operation represented by the substitution $T^{-1}ST$ is called the *transformation* of S by T, and the resulting substitution the *conjugate* of S with respect to T.

Any substitution is similar to its conjugate with respect to any *other substitution.* To prove this, let S be transformed by the substitution

$$T \equiv \begin{pmatrix} abc & . & . & l & . & . & . \\ a'b'c' & . & . & l' & . & . & . \end{pmatrix},$$

and let $(abc\ .\ .\ .\ l)$ be one of the cycles of S. The effect of the operation T^{-1} is to replace a' by a, which by the operation of S is replaced by b, which again by the operation T is replaced by b'. The substitution $T^{-1}ST$ therefore replaces a' by b', b' by c', $.\ .\ .\ l'$ by a'; and to the cycle $(abc\ .\ .\ .\ l)$ in S corresponds the cycle $(a'b'c'\ .\ .\ .\ l')$ in its conjugate.

The transformation of S by T is completed by replacing in each cycle of S every letter by that which stands under it in the substitution T. The resulting substitution is therefore similar to S. Reciprocally, it is clear that, if two substitutions S_1 and S_2 are similar, a substitution T can be found which transforms one into the other.

The products ST and TS, which are in general different, are always *similar*, since $ST = T^{-1}(TS)\,T$.

The conjugate of the product ST with respect to a third substitution U is equal to the product of the conjugates of its factors, for we have $U^{-1}(ST)\,U = U^{-1}SUU^{-1}TU$.

If two substitutions S, T are commutative, their conjugates with respect to U are commutative, for if $ST = TS$, we have

$$U^{-1}SUU^{-1}TU = U^{-1}TUU^{-1}SU.$$

Section II.—Multiple-valued Functions and Groups.

224. Definition of Group. Symmetric Group.—According to the number of values a function of $x_1, x_2, \ldots x_n$ assumes under the operation of the N possible substitutions, it is said to be one-valued, two-valued, . . . ρ-valued. A symmetric function of these elements, being unaltered by any transposition (Art. 27), and therefore by any product of transpositions, is a one-valued function. If a function be not symmetric, it has two or more values which may be derived from the one supposed given by the process of substitution. Consider, for example, the two rational functions of three elements—

$$\Phi_1 \equiv x_1^2x_2 + x_2^2x_3 + x_3^2x_1, \quad \sqrt{\Delta} \equiv (x_1 - x_2)(x_1 - x_3)(x_2 - x_3).$$

Each of these is two-valued. Of the six possible substitutions, viz.,

$$1, \quad (123), \quad (132), \quad (12), \quad (13), \quad (23),$$

the first three leave Φ_1 unaltered, while by each of the last three it is changed into its second value $x_2^2x_1 + x_3^2x_2 + x_1^2x_3 \equiv \Phi_2$. In the same way $\sqrt{\Delta}$ also is unaltered by the first three, and is changed by the last three into its second value $-\sqrt{\Delta}$. As an example in the case of four elements, consider the function

$$\phi_1 \equiv x_1x_2 + x_3x_4.$$

There are, in addition to ϕ_1, two other values, viz.,

$$\phi_2 \equiv x_1x_3 + x_2x_4 \quad \text{and} \quad \phi_3 \equiv x_1x_4 + x_2x_3;$$

the function is therefore three-valued.

It can be easily verified that ϕ_1 is unaltered by the following eight substitutions :—

$$1, \ (12), \ (34), \ (12)(34), \ (13)(24), \ (14)(23), \ (1324), \ (1423),$$

and that any of the remaining sixteen will change ϕ_1 into one or other of the two remaining values. The substitutions which leave a function unchanged constitute a *group*. It is clear that any combination by multiplication of two or more members of the group will itself be a substitution contained in the group. We give therefore the formal definition of a group as follows :—

A system of distinct substitutions is said to form a group when all powers and products of these substitutions form part of the same system.

The number of substitutions contained in a group is called the *order* of the group.

The number of elements operated on is the *degree* of the group.

The group which leaves a function $\phi(x_1, x_2, \ldots x_n)$ unaltered is called the *group belonging to* ϕ, or, briefly, the *group of* ϕ.

The total number of N substitutions, of course, constitutes a group. This is called the *symmetric group*, since all its members leave any symmetric function unaltered.

One group may contain all the substitutions of another in addition to others peculiar to itself. In such a case the included group is called a *sub-group* of the former.

The symmetric group contains every other group as a sub-group. Any substitution whatever, with all its distinct powers, constitute a group contained as a sub-group in the symmetric group. Next in importance to the symmetric is the alternate group, which we will now define.

225. **Alternate Group.**—Let us consider in the case of n elements the rational function

$$\begin{aligned} \Pi(x_i - x_j) = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4) \ldots (x_1 - x_n) \\ (x_2 - x_3)(x_2 - x_4) \ldots (x_2 - x_n) \\ (x_3 - x_4) \ldots (x_3 - x_n) \\ \ldots \ldots \\ (x_{n-1} - x_n), \end{aligned}$$

consisting of the product of all the differences of the elements. The square of Π is the well-known symmetric function, the discriminant Δ: and therefore Π has two values equal numerically with opposite signs, viz. $\sqrt{\Delta}$ and $-\sqrt{\Delta}$. Such two-valued functions are called *alternating functions*. It is clear that any transposition alters the sign of Π, for consider the transposition (x_α, x_β), and rearrange Π, without altering its value, so that x_α, x_β occupy the same position as x_1, x_2, the sign in front being not necessarily positive. In the new form for Π the transposition (x_α, x_β) alters the sign of the first factor in the upper row, and interchanges the remaining factors of the upper row with the factors of the second row. It does not affect any of the factors in the remaining rows; hence the sign of the product is altered. This follows also obviously by expressing Π as a determinant whose first row is $x_1^{n-1}, x_1^{n-2}, x_1^{n-3}, \ldots x_1^{l}$, and whose other rows are similarly formed in order from $x_2, x_3, \ldots x_n$. Any second transposition restores the original sign; hence the effect of the product of two, or any even number of, transpositions is to leave $\sqrt{\Delta}$ unaltered, and the effect of the product of any odd number is to change $\sqrt{\Delta}$ into its second value $-\sqrt{\Delta}$, or $-\sqrt{\Delta}$ into its second value $\sqrt{\Delta}$.

A substitution can be expressed in many different ways as a product of transpositions, but, however variously expressed, *the number of such factors must be always even or always odd;* for it is clear that the same substitution cannot at the same time change the sign of $\sqrt{\Delta}$ and leave it unchanged. Since the product of two even substitutions is itself an even substitution, it follows that unity, along with all substitutions which are made up of an even number of transpositions, constitutes a group, and that $\sqrt{\Delta}$ and $-\sqrt{\Delta}$ are both functions belonging to this group. It is called the *alternate group:* we proceed to investigate its order. Let the alternate group of n elements consist of the following substitutions:—

$$S_1 = 1, \quad S_2, \quad S_3, \quad . \quad . \quad . \quad . \quad . \quad S_r, \qquad (1)$$

and let the remaining substitutions of the symmetric group, all consisting of an odd number of transpositions, and therefore distinct from the former, be

$$S_1', \quad S_2', \quad S_3', \quad . \quad . \quad . \quad . \quad S_t'. \qquad (2)$$

We select now any transposition T, and form by multiplication the two following series:—

$$S_1T, \quad S_2T, \quad S_3T, \quad . \quad . \quad . \quad . \quad S_rT, \qquad (3)$$

$$S_1'T, \quad S_2'T, \quad S_3'T, \quad . \quad . \quad . \quad . \quad S_r'T. \qquad (4)$$

Every substitution in (3) is composed of an odd number of transpositions, and is therefore contained in (2), and every substitution in (4) of an even number, and is therefore contained in (1). It follows that $r \leqq t$, and also $r \geqq t$; hence $r = t$; and since $r + t = N$, we have finally for the order of the alternate group

$$r = \tfrac{1}{2}N.$$

226. **Conjugate values of Multiple-valued Functions and Conjugate Groups. Theorem.**—*The order of any group is an exact divisor of N, the quotient being the number of distinct values of the corresponding multiple-valued function.*

In establishing this important theorem it is convenient first to find a function ϕ_1 which is unaltered by all the substitutions $S_1 \equiv 1, S_2, S_3, \ldots S_r$ of the group G_1, whose order is r and degree n. To find such a function, we take a function $\chi_1 = x_1{}^a x_2{}^b x_3{}^c \ldots x_n{}^l$, where $a, b, c, \ldots l$ are all different integers, and therefore χ_1 assumes N different values for all the substitutions of the symmetric group. Taking $\chi_2 = S_2\chi_1$, $\chi_3 = S_3\chi_1$, and so on, we shall prove that $\phi_1 = \chi_1 + \chi_2 + \ldots + \chi_r$ is unaltered by the substitutions of G_1. Equally well we might take ϕ_1 equal to the sum of any powers of $\chi_1, \chi_2, \ldots \chi_r$, which would be a particular case of taking a different set of integers for $a, b, c, \ldots l$. Taking, then, $\phi_1 = \chi_1 + \chi_2 + \ldots \chi_r = (S_1 + S_2 + \ldots S_r)\chi_1$, if we multiply by any substitution S_α of G_1, we get $S_\alpha\phi_1 = (S_1S_\alpha + S_2S_\alpha + \ldots S_rS_\alpha)\chi_1$. Now, if S_β, S_γ are substitutions of G_1, $S_\beta S_\alpha$ is by hypothesis a substitution of G_1, and moreover $S_\beta S_\alpha\chi_1$ is not equal to $S_\gamma S_\alpha\chi_1$, for if it were, multiplying by $S_\alpha{}^{-1}$, $S_\beta\chi_1 = S_\gamma\chi_1$, and therefore as χ_1 has N values, $S_\beta = S_\gamma$. The effect, then, of multiplying $\chi_1, \chi_2, \ldots \chi_r$ by S_α is to reproduce them all in some order, and accordingly ϕ_1 is unaltered by any substitution of G_1. Any substitution T which does not belong to G_1, alters ϕ_1. For if $T\phi_1 \equiv \phi_1$, we must have $T\chi_\alpha \equiv \chi_\beta$, $\therefore S_\alpha T\chi_1 \equiv S_\beta\chi_1$, $\therefore S_\alpha T \equiv S_\beta$ as χ_1 has N distinct values, $\therefore T \equiv S_\alpha{}^{-1}S_\beta$, $\therefore T$ belongs to the group G_1.

To proceed now with the proof of the general theorem, let Σ_2 be a substitution not contained in G_1 and which therefore alters ϕ_1 to a different value ϕ_2.

Multiplying the members of G_1 by Σ_2, we have the following series of substitutions all belonging, of course, to the symmetric group :—

$$S_1\Sigma_2,\quad S_2\Sigma_2,\quad S_3\Sigma_2,\ \ldots\ S_r\Sigma_2.$$

The members of this series have the following properties :—(1) they are all distinct from one another, for if $S_\alpha\Sigma_2 = S_\beta\Sigma_2$, multiplying by Σ_2^{-1}, we get $S_\alpha = S_\beta$; (2) they all change ϕ_1 to ϕ_2, for S_α leaves ϕ_1 unaltered and Σ_2 changes it to ϕ_2; and (3) there are no other substitutions in the symmetric group possessing this property, for if T is any substitution changing ϕ_1 to ϕ_2, $T\Sigma_2^{-1}$ leaves ϕ_1 unaltered and therefore belongs to G_1; hence $T\Sigma_2^{-1} = S_\alpha$, and $\therefore T = S_\alpha\Sigma_2$.

We now form another row by means of a substitution Σ_3 not contained in G_1 or the series $S_\alpha\Sigma_2$. Such a substitution will alter ϕ_1 to a new value ϕ_3, for if it leaves ϕ_1 unaltered it belongs to G_1, and if it alters ϕ_1 to ϕ_2 it will be in the second row. Proceeding in this way by selecting at each new stage a value for Σ not contained in the rows previously obtained, we exhaust the N substitutions of the symmetric group, arranging them in a table of ρ rows, associated with which are values of ϕ, viz. $\phi_1, \phi_2, \ldots \phi_\rho$ which are all different, and moreover contain all the values of ϕ, as the table indicates the effect of any substitution on ϕ_1. We thus obtain the following table, in which Σ_1 is written for symmetry instead of 1 :—

$$\begin{array}{lllll}
S_1\Sigma_1, & S_2\Sigma_1, & S_3\Sigma_1, & \ldots & S_r\Sigma_1,\\
S_1\Sigma_2, & S_2\Sigma_2, & S_3\Sigma_2, & \ldots & S_r\Sigma_2,\\
S_1\Sigma_3, & S_2\Sigma_3, & S_3\Sigma_3, & \ldots & S_r\Sigma_3,\\
\cdot & \cdot & \cdot & \cdot & \cdot\\
S_1\Sigma_\rho, & S_2\Sigma_\rho, & S_3\Sigma_\rho, & \ldots & S_r\Sigma_\rho.
\end{array}$$

This arrangement of the substitutions of the symmetric group by means of the substitutions of G_1 might be established without any reference to ϕ_1 by noting that if, say, Σ_3 is not contained in the previous rows, then $S_\alpha\Sigma_3$ is not equal to S_β or $S_\beta\Sigma_2$. For if $S_\alpha\Sigma_3 = S_\beta$, $\Sigma_3 = S_\alpha^{-1}S_\beta$, and $\therefore \Sigma_3$ is in the first row; and if $S_\alpha\Sigma_3 = S_\beta\Sigma_2$, $\Sigma_3 = S_\alpha^{-1}S_\beta\Sigma_2$, and therefore is in the second row.

Similarly when we take a new substitution Σ_4 of the symmetric group not contained in the first three rows, we obtain a new row none of whose members are in the first three rows; for if $S_\alpha\Sigma_4 = S_\beta$ or $S_\beta\Sigma_2$ or $S_\beta\Sigma_3$, then $\Sigma_4 = S_\alpha^{-1}S_\beta$ or $S_\alpha^{-1}S_\beta\Sigma_2$ or $S_\alpha^{-1}S_\beta\Sigma_3$, and therefore Σ_4 is in the first or second or third row. Proceeding in this we exhaust all the N substitutions of the symmetric group, arranging them in the above table which consists of ρ lines of r each, and hence it follows that $r\rho = N$, and the theorem is proved.

On account of the similarity of the different values

$$\phi_1, \phi_2, \ldots \phi_k \ldots \phi_r$$

of the ρ-valued function ϕ, it is evident, *a priori*, that each of these functions will have a group similar to the group of ϕ_1. It can be readily shown that the group of any function ϕ_k is obtained by *transforming* (Art. 223) all the substitutions of G by the substitution Σ_k which alters ϕ_1 to ϕ_k. In fact, any substitution $\Sigma_k^{-1}S_\alpha\Sigma_k$ leaves ϕ_k unaltered, for Σ_k^{-1} changes it to ϕ_1, which is unaltered by S_α, and consequently changed by Σ_k to ϕ_k. The group of ϕ_k is therefore

$$\Sigma_k^{-1}S_1\Sigma_k, \quad \Sigma_k^{-1}S_2\Sigma_k, \quad \Sigma_k^{-1}S_3\Sigma_k, \quad \ldots \quad \Sigma_k^{-1}S_r\Sigma_k,$$

where each substitution of G_1 is transformed by Σ_k. This result may be represented briefly by the notation

$$G_k = \Sigma_k^{-1}G_1\Sigma_k.$$

$G_1, G_2, G_3, \ldots G_\rho$ are called *conjugate groups*, and the corresponding functions $\phi_1, \phi_2, \phi_3, \ldots \phi_\rho$ *conjugate functions*.

It is clear (Art. 223) that *any two conjugate groups consist of similar substitutions.*

What is proved above as to the relation between the orders of G_1 and the symmetric group is true, more generally, of the relation between the orders of G_1 and any wider group G_1' in which G_1 is contained as a sub-group; that is to say, *the order r of G_1 is an exact divisor of the order r' of G_1' which contains G_1 as a sub-group, the quotient m being the number of distinct values of a multiple-valued function unaltered by the substitutions of G_1, which are obtained by the substitutions of G_1'.*

The proof, which is similar to that given above, consists in arranging the r' substitutions of G_1' in m rows, of which the first is made up of the substitutions of G_1, the second of these substitutions multiplied by Σ_2, a substitution of G_1' not contained in G_1, the third by a substitution Σ_3 of G_1' not contained in the first two rows, and so on until all the substitutions of G_1' are exhausted. The table shows the effect of every substitution of G_1' on ϕ_1, either leaving it unaltered or altering it to ϕ_1 or ϕ_2 or . . . or ϕ_m; so that in fact $\phi_1 + \phi_2 + \ . \ . \ . \ + \phi_m$ is a function unaltered by the substitutions of G_1'. We thus obtain, in addition to $r\rho = r'\rho' = N$, that $r' = mr$ and $\therefore\ \rho = m\rho'$.

Examples.

1. Construct, for four elements, the conjugate groups corresponding to the different values of the function $\phi_1 \equiv x_1x_2 + x_3x_4$.

It is easily seen that there are only three distinct values of this function, viz.,

$$\phi_1 \equiv x_1x_2 + x_3x_4, \quad \phi_2 \equiv x_1x_3 + x_2x_4, \quad \phi_3 \equiv x_1x_4 + x_2x_3,$$

and each has therefore a group of order 8.

The group of ϕ_1 consists of the following eight substitutions:—

$$G_1 \equiv [1, \quad (12), \quad (34), \quad (12)(34), \quad (13)(24), \quad (14)(23), \quad (1324), \quad (1423)].$$

If we take any substitution, e.g. (23), which changes ϕ_1 to ϕ_2, and any other, say (24), which changes ϕ_1 to ϕ_3, and form the table of the foregoing Article, we obtain all the twenty-four substitutions of the symmetric group as follows:—

1	(12)	(34)	(12)(34)	(13)(24)	(14)(23)	(1324)	(1423)
(23)	(132)	(234)	(1342)	(1243)	(14)	(124)	(143)
(24)	(142)	(243)	(1432)	(13)	(1234)	(134)	(123)

The first row is the group G_1; the other rows not constituting groups, but being such that the members of the second (and no others) all convert ϕ_1 into ϕ_2, and the members of the third (and no others) all convert ϕ_1 into ϕ_3. The group G_2 corresponding to ϕ_2 is obtained by *transforming* the substitutions of G_1 by (23), and this is done by simply interchanging 2 and 3 in the substitutions of G_1. In this way we find easily the groups of ϕ_2 and ϕ_3, as follows:—

$$G_2 \equiv [1, \quad (13), \quad (24), \quad (13)(24), \quad (12)(34), \quad (14)(23), \quad (1234), \quad (1432)].$$

$$G_3 \equiv [1, \quad (14), \quad (23), \quad (14)(23), \quad (13)(24), \quad (12)(34), \quad (1342), \quad (1243)].$$

It will be observed that none of the circular substitutions of the 3rd order are present in any of these groups, and the three groups have certain substitutions common. In fact, the substitution unity must be common to all conjugate groups; and here G_1, G_2, G_3 have the three substitutions (12) (34), (13) (24), (14) (23) common, in addition to unity, these four substitutions forming a common sub-group of the three conjugate groups.

2. Verify that the substitutions of G_1 in the preceding example form a closed group; that is to say, any multiplication of two of its members always reproduces some member of the group.

Representing the substitutions of G_1 in the order of the preceding example by the symbols 1, A, B, C, D, E, F, G, we have the following multiplication table, which the students will easily verify :—

	1	A	B	C	D	E	F	G
$1 \equiv 1$	1	A	B	C	D	E	F	G
$(12) \equiv A$	A	1	C	B	G	F	E	D
$(34) \equiv B$	B	C	1	A	F	G	D	E
$(12)(34) \equiv C$	C	B	A	1	E	D	G	F
$(13)(24) \equiv D$	D	F	G	E	1	C	A	B
$(14)(23) \equiv E$	E	G	F	D	C	1	B	A
$(1324) \equiv F$	F	D	E	G	B	A	C	1
$(1423) \equiv G$	G	E	D	F	A	B	1	C

In effecting the multiplication, the factor from the first column is to be placed at the *left-hand side of each symbol of the upper row in turn.*

It will be observed that G_1 contains the sub-groups

$$[1, A, B, C], \quad [1, C, D, E], \quad [1, C, F, G],$$

all of order 4, and several also of order 2, e.g. $[1, A]$, $[1, C]$.

3. Construct the alternate group G' for four elements. The substitutions, which consist of an even number of transpositions, can easily be selected from the twenty-four given in Ex. 1. They are, in fact, the four substitutions

1, (12) (34), (13) (24), (14) (23), along with the eight circular substitutions of the third order. These we arrange in three rows, as follows :—

$$G' \equiv \begin{cases} 1 & (12)(34) & (13)(24) & (14)(23), \\ (132) & (234) & (124) & (143), \\ (142) & (243) & (134) & (123). \end{cases}$$

To this group belongs the function $\sqrt{\Delta}$. If each substitution be multiplied by any transposition, say (23), which changes $\sqrt{\Delta}$ to $-\sqrt{\Delta}$, the remaining twelve substitutions of the symmetric group are obtained. If each member of G' be *transformed* by (23), we obtain the group of $-\sqrt{\Delta}$. It is easily verified that this coincides with G' the group of $\sqrt{\Delta}$. For example, (12) (34) becomes (13) (24) by this transformation; (14) (23) is unaltered; (123) and (132) are interchanged; and so on. The two conjugate groups therefore coincide in this case, $\sqrt{\Delta}$ and $-\sqrt{\Delta}$ both belonging to the same group. The same is true for any number of elements (Art. 225).

The arrangement in three rows of the substitutions of G' illustrates what is proved at the conclusion of the foregoing Article. The four substitutions in the first row form a sub-group of G'; the four in the second row are obtained from these by multiplication (on the right-hand side) by (132), and the last four by multiplication by (142); the order 4 of the sub-group being a divisor of the order of G'. To this group, which we will call H, viz.,

$$H \equiv [1, \quad (12)(34), \quad (13)(24), \quad (14)(23)],$$

belongs the function

$$A(x_1x_2 + x_3x_4) + B(x_1x_3 + x_2x_4) + C(x_1x_4 + x_2x_3),$$

in which A, B, C are any arbitrary constants.

This function has six distinct values for the substitutions of the symmetric group, viz.,

$$A\phi_1 + B\phi_2 + C\phi_3, \quad A\phi_2 + B\phi_3 + C\phi_1, \quad A\phi_3 + B\phi_1 + C\phi_2,$$
$$A\phi_1 + B\phi_3 + C\phi_2, \quad A\phi_2 + B\phi_1 + C\phi_3, \quad A\phi_3 + B\phi_2 + C\phi_1.$$

These have all the same group H, the six conjugate groups coinciding in this case; in fact, any transformation of the symmetric group operated upon the substitutions of H will reproduce the same four in some order. Such a group is called an *invariant sub-group* of the symmetric group. The alternate group is also an invariant sub-group.

4. Prove that the group derived from the $n-1$ transpositions (12), (13), . . . $(1n)$, is identical with the symmetric group.

Every substitution, being expressible by transpositions, can be represented as a product of members of this series (Ex. 12, Art. 222).

5. Prove, for any number of elements, that there is only one group of order $\frac{1}{2}N$, viz., the alternate group.

Let the group of order $\frac{1}{2}N$ be (1)

$$S_1 \equiv 1, \quad S_2, \quad S_3, \ . \ . \ . \ S_{\frac{1}{2}N}$$

Multiplying this, first at the left side and afterwards at the right, by any substitution T of the symmetric group not already contained in it, we have the two series

$$T, \quad TS_2, \quad TS_3, \ldots TS_{\frac{1}{2}N} \tag{2}$$
$$T, \quad S_2T, \quad S_3T, \ldots S_{\frac{1}{2}N}T. \tag{3}$$

Each of these must consist of the $\frac{1}{2}N$ substitutions not contained in (1); hence the two series are identical, and whatever i may be, we have for some value of j the relation

$$TS_i = S_jT, \text{ or } S_i = T^{-1}S_jT;$$

and since also $S_\beta = S_\alpha^{-1}S_\gamma S_\alpha$, it follows that for any substitution T whatsoever $S_i = T^{-1}S_jT$, and hence the group (1) contains all substitutions similar to any one contained in it. Hence (1) cannot comprise any single transposition, for if it did it would contain all such, and be consequently identical with the symmetric group (Ex. 4).

It can now be shown that (1) contains as a substitution the product of any pair of transpositions. For this purpose, suppose T in the series (2) to be any transposition. The effect of multiplying both (1) and (2) by any second transposition U is to interchange the two series (1) and (2). It is proved therefore that UT must be one of the substitutions of (1), as $S_1 = 1$ is one of them.

From this it appears that every two-valued function belongs to the alternate group, since this is the only group whose order satisfies the equation $2r = N$.

6. The alternate group includes all circular substitutions of odd order, and none of even order.

7. Prove that a group which contains all the circular substitutions of the third order is either the alternate or the symmetric group.

Use Ex. 13, Art. 222.

8. Prove that a group which contains all the circular substitutions of the fifth order contains also all of the third order. For

$$(acdeb)\,(acbed) \equiv (abc).$$

9. The order of a group is a multiple of the order of any one of the substitutions of the group.

10. If n is a prime number, every group of order n is composed of n powers of a circular substitution of order n.

11. If two groups have common substitutions, these themselves form a group, and their number is a common divisor of the orders of both groups.

12. If the members of a group are all transformed by the same substitution, the conjugates thus derived themselves form a group.

Use the relations given at the end of Art. 223.

227. **Formation of Functions of a given Group. The Galois Function.**—We take up again the problem of finding rational integral functions of n variables, $x_1, x_2, \ldots x_n$, which remain unaltered by the substitutions of a group G_1, which we dealt with at the beginning of Art. 226. We select for ψ_1 the following different type of function having N distinct values for all the substitutions of the symmetric group :—

$$\psi_1 = a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n,$$

in which $a_1, a_2, \ldots a_n$ are n distinct arbitrary constants. This function is called the *Galois Function*. As in Art. 226, obtaining $\psi_2, \psi_3, \ldots \psi_r$ by the substitutions of G_1, any symmetric function of $\psi_1, \psi_2, \ldots \psi_r$ will be unaltered by the substitutions of G_1. In particular, $\phi_1 = (y + \psi_1)(y + \psi_2) \ldots (y + \psi_r)$ will be unaltered by the substitutions of G_1 and altered to a different value by any substitution not contained in G_1. The function ϕ_1 is not therefore unaltered by the substitutions of any wider group containing G_1 as a sub-group. Expanding ϕ_1 in powers of y, although some of the coefficients of the powers of y may be unaltered by the substitutions of a wider group, all are not unaltered, and so one of them will provide a function unaltered by the substitutions of G_1 and altered by those of any wider group. Noting the forms of expression of the sum of the powers of the roots of an equation in terms of the coefficients, we see that instead of the coefficients of the powers of y in ϕ_1 we may take sums of powers of $\psi_1, \psi_2, \ldots \psi_r$ up to the r^{th}, and deduce that one at least of such r sums of powers will provide a function unaltered by the substitutions of G_1 and altered by any other substitution of the symmetric group. We add a few simple examples to illustrate modes of finding functions of a given group.

EXAMPLES.

1. Form a function of three variables which shall be unchanged by all the substitutions of the alternate group, viz.,

$$[1, \quad (123), \quad (132)].$$

Multiplying this, first at the left side and afterwards at the right, by any substitution T of the symmetric group not already contained in it, we have the two series

$$T, \quad TS_2, \quad TS_3, \ldots TS_{\frac{1}{2}N} \tag{2}$$

$$T, \quad S_2T, \quad S_3T, \ldots S_{\frac{1}{2}N}T. \tag{3}$$

Each of these must consist of the $\frac{1}{2}N$ substitutions not contained in (1); hence the two series are identical, and whatever i may be, we have for some value of j the relation

$$TS_i = S_jT, \text{ or } S_i = T^{-1}S_jT;$$

and since also $S_\beta = S_\alpha^{-1}S_\gamma S_\alpha$, it follows that for any substitution T whatsoever $S_i = T^{-1}S_jT$, and hence the group (1) contains all substitutions similar to any one contained in it. Hence (1) cannot comprise any single transposition, for if it did it would contain all such, and be consequently identical with the symmetric group (Ex. 4).

It can now be shown that (1) contains as a substitution the product of any pair of transpositions. For this purpose, suppose T in the series (2) to be any transposition. The effect of multiplying both (1) and (2) by any second transposition U is to interchange the two series (1) and (2). It is proved therefore that UT must be one of the substitutions of (1), as $S_1 = 1$ is one of them.

From this it appears that every two-valued function belongs to the alternate group, since this is the only group whose order satisfies the equation $2r = N$.

6. The alternate group includes all circular substitutions of odd order, and none of even order.

7. Prove that a group which contains all the circular substitutions of the third order is either the alternate or the symmetric group.

Use Ex. 13, Art. 222.

8. Prove that a group which contains all the circular substitutions of the fifth order contains also all of the third order. For

$$(acdeb)\,(acbed) \equiv (abc).$$

9. The order of a group is a multiple of the order of any one of the substitutions of the group.

10. If n is a prime number, every group of order n is composed of n powers of a circular substitution of order n.

11. If two groups have common substitutions, these themselves form a group, and their number is a common divisor of the orders of both groups.

12. If the members of a group are all transformed by the same substitution, the conjugates thus derived themselves form a group.

Use the relations given at the end of Art. 223.

227. Formation of Functions of a given Group. The Galois Function.—We take up again the problem of finding rational integral functions of n variables, $x_1, x_2, \ldots x_n$, which remain unaltered by the substitutions of a group G_1, which we dealt with at the beginning of Art. 226. We select for ψ_1 the following different type of function having N distinct values for all the substitutions of the symmetric group :—

$$\psi_1 = a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n,$$

in which $a_1, a_2, \ldots a_n$ are n distinct arbitrary constants. This function is called the *Galois Function*. As in Art. 226, obtaining $\psi_2, \psi_3, \ldots \psi_r$ by the substitutions of G_1, any symmetric function of $\psi_1, \psi_2, \ldots \psi_r$ will be unaltered by the substitutions of G_1. In particular, $\phi_1 = (y + \psi_1)(y + \psi_2) \ldots (y + \psi_r)$ will be unaltered by the substitutions of G_1 and altered to a different value by any substitution not contained in G_1. The function ϕ_1 is not therefore unaltered by the substitutions of any wider group containing G_1 as a sub-group. Expanding ϕ_1 in powers of y, although some of the coefficients of the powers of y may be unaltered by the substitutions of a wider group, all are not unaltered, and so one of them will provide a function unaltered by the substitutions of G_1 and altered by those of any wider group. Noting the forms of expression of the sum of the powers of the roots of an equation in terms of the coefficients, we see that instead of the coefficients of the powers of y in ϕ_1 we may take sums of powers of $\psi_1, \psi_2, \ldots \psi_r$ up to the r^{th}, and deduce that one at least of such r sums of powers will provide a function unaltered by the substitutions of G_1 and altered by any other substitution of the symmetric group. We add a few simple examples to illustrate modes of finding functions of a given group.

EXAMPLES.

1. Form a function of three variables which shall be unchanged by all the substitutions of the alternate group, viz.,

$$[1, \quad (123), \quad (132)].$$

Operating with these substitutions on the Galois function, we have

$$\psi_1 \equiv \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3,$$
$$\psi_2 \equiv \alpha_1 x_2 + \alpha_2 x_3 + \alpha_3 x_1,$$
$$\psi_3 \equiv \alpha_1 x_3 + \alpha_2 x_1 + \alpha_3 x_2.$$

$\Sigma\psi_1$ and $\Sigma\psi_1^2$ are both symmetric in x_1, x_2, x_3. But by means either of $\Sigma\psi_1^3$ or $\Sigma\psi_1^4$, we can obtain the unsymmetric functions

$$x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_1 \quad \text{and} \quad x_1^2 x_3 + x_2^2 x_1 + x_3^2 x_2,$$

both of which must belong to the given group. If these functions be called Φ_x and Φ_x', it is, in fact, easily verified that

$$\Sigma\psi_1^3 = \Sigma\alpha_1^3 \Sigma x_1^3 + 6\alpha_1\alpha_2\alpha_3 x_1 x_2 x_3 + 3(\Phi_x \Phi_\alpha + \Phi_x' \Phi_\alpha'),$$

where $\Phi_\alpha = \alpha_1^2\alpha_2 + \alpha_2^2\alpha_3 + \alpha_3^2\alpha_1, \quad \Phi_\alpha' = \alpha_1^2\alpha_3 + \alpha_2^2\alpha_1 + \alpha_3^2\alpha_2.$

The result is more readily obtained by using the method of Art. 230 and taking [illegible]

2. Investigate functions of four variables which shall belong to the group

$$H = [1, \quad (12)(34), \quad (13)(24), \quad (14)(23)].$$

Writing down the values of the four Galois functions as follows :—

$$\psi_1 \equiv \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4,$$
$$\psi_2 \equiv \alpha_1 x_2 + \alpha_2 x_1 + \alpha_3 x_4 + \alpha_4 x_3,$$
$$\psi_3 \equiv \alpha_1 x_3 + \alpha_2 x_4 + \alpha_3 x_1 + \alpha_4 x_2,$$
$$\psi_4 \equiv \alpha_1 x_4 + \alpha_2 x_3 + \alpha_3 x_2 + \alpha_4 x_1,$$

we find that $\Sigma\psi_1$ is symmetric in x_1, x_2, x_3, x_4, but that $\Sigma\psi_1^2$ is not so. From the latter we readily obtain the function $A\phi_1 + B\phi_2 + C\phi_3$ of Ex. 3, Art. 226, ϕ_1, ϕ_2, ϕ_3 representing the same functions of x_1, x_2, x_3, x_4, as in Ex. 1 of the Article referred to. We have, in fact,

$$\Sigma\psi_1^2 = \Sigma\alpha_1^2 \Sigma x_1^2 + 4(\alpha_1\alpha_2 + \alpha_3\alpha_4)\phi_1 + 4(\alpha_1\alpha_3 + \alpha_2\alpha_4)\phi_2 + 4(\alpha_1\alpha_4 + \alpha_2\alpha_3)\phi_3.$$

The unsymmetric functions occurring here, viz. ϕ_1, ϕ_2, ϕ_3, belong respectively to the wider groups G_1, G_2, G_3 of order eight. The sum of these with arbitrary coefficients belongs to the given group H, and is a six-valued function.

3. Investigate functions of four variables for the group

$$G_1 = \{1, \ (12), \ (34), \ (12)(34), \ (13)(24), \ (14)(23), \ (1324), \ (1423)\}.$$

Taking, along with the four values of ψ in the preceding example, the additional four

$$\psi_5 \equiv \alpha_1 x_2 + \alpha_2 x_1 + \alpha_3 x_3 + \alpha_4 x_4,$$
$$\psi_6 \equiv \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_4 + \alpha_4 x_3,$$
$$\psi_7 \equiv \alpha_1 x_3 + \alpha_2 x_4 + \alpha_3 x_2 + \alpha_4 x_1,$$
$$\psi_8 \equiv \alpha_1 x_4 + \alpha_2 x_3 + \alpha_3 x_1 + \alpha_4 x_2,$$

we easily verify the following relation :—

$$\Sigma\psi_1^2 = 2\Sigma\alpha_1^2\Sigma x_1^2 + 8(\alpha_1\alpha_2 + \alpha_3\alpha_4)(x_1x_2 + x_3x_4) + 2(\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4)(x_1 + x_2)(x_3 + x_4);$$

whence the functions $x_1x_2 + x_3x_4$, and $(x_1 + x_2)(x_3 + x_4)$ are obtained, both belonging to the given group, since there is no wider group except the symmetric in which G_1 is contained as a sub-group.

It is clear that this method may be used to discover, by means of the symmetric functions of higher orders, an infinite variety of functions corresponding to a given group.

228. **Theorem.**—*Every integral symmetric function of the distinct values of any integral multiple-valued function of n elements is a symmetric function of the elements themselves.*

Although this proposition appears sufficiently evident from the similarity of structure of the conjugate values $\phi_1, \phi_2, \phi_3, \ldots \phi_\rho$ of a ρ-valued function (Art. 226), we may give a formal proof as follows. Let $F(\phi_1, \phi_2, \ldots \phi_\rho)$ be any rational integral symmetric function of the ρ-values. Any substitution whatever S (affecting the elements) applied to these ρ-values either leaves any function unchanged or replaces it by one of the others. No two of the resulting values can be equal, for if $S\phi_i$ were equal to $S\phi_j$, it would follow, by applying the substitution S^{-1}, that $\phi_i = \phi_j$, which is contrary to hypothesis. Consequently the same ρ values of ϕ are reproduced by S in some order or other. The symmetric function F therefore remains unchanged by any substitution, and is consequently a symmetric function of the elements themselves.

From this is derived immediately the following corollary :—

Cor.—*The ρ distinct values of any integral multiple-valued function are roots of an equation whose coefficients are integral symmetric functions of the elements themselves.*

For an example of this we refer to Ex. 4, Art. 39, Vol. I. What is here proved with regard to rational integral functions can be readily extended to all rational functions, whether integral or not; for any fraction may be converted by the method of Art. 194 into an equivalent form whose denominator is symmetric in the elements.

229. **Theorem.**—*Two functions belonging to the same group can be rationally expressed each in terms of the other.*

This important proposition, to which we now apply the principles of the method of substitution, has been discussed before (Art. 194) from a somewhat different point of view. Let ϕ_1 and ψ_1 be two functions belonging to the same group

$$G_1 \equiv [1, \quad S_2, \quad S_3, \ . \ . \ . \ S_r],$$

of order r and degree n, each of these functions having ρ distinct values, where $r\rho = N$. Any substitution not contained in G_1 will convert ϕ_1 into another of its values, say ϕ_k, and at the same time ψ_1 into ψ_k. By operating all possible substitutions, ρ pairs of values ϕ_1, ψ_1; ϕ_2, ψ_2; . . . ϕ_ρ, ψ_ρ are obtained. Now, in the first place, the rational function

$$\Sigma\phi^i\psi^j = \phi_1{}^i\psi_1{}^j + \phi_2{}^i\psi_2{}^j + \ . \ . \ . \ + \phi_k{}^i\psi_k{}^j + \ . \ . \ . \ + \phi_\rho{}^i\psi_\rho{}^j \quad (1)$$

is clearly a symmetric function of the elements, for it appears, by the same reasoning as that of the preceding Article, that any substitution whatever affecting the elements will reproduce in some order the terms of this sum, viz. $\Sigma\phi^i\psi^j$, which is therefore a symmetric function of the elements. If, now, we take $j = 1$, and assign to i all the values $0, 1, 2, \ . \ . \ . \ . \ \rho - 1$ in succession, we obtain the following ρ equations linear in $\psi_1, \psi_2, \ . \ . \ . \ \psi_\rho$:—

$$\left.\begin{aligned} \psi_1 + \quad \psi_2 + \ . \ . \ . \ + \quad \psi_\rho &= T_0 \\ \phi_1\psi_1 + \quad \phi_2\psi_2 + \ . \ . \ . \ + \quad \phi_\rho\psi_\rho &= T_1 \\ \phi_1{}^2\psi_1 + \quad \phi_2{}^2\psi_2 + \ . \ . \ . \ + \quad \phi_\rho{}^2\psi_\rho &= T_2 \\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ \phi_1{}^{\rho-1}\psi_1 + \phi_2{}^{\rho-1}\psi_2 + \ . \ . \ . \ + \phi_\rho{}^{\rho-1}\psi_\rho &= T_{\rho-1} \end{aligned}\right\}, \qquad (2)$$

where $T_0, T_1, T_2, \ . \ . \ .$ are all symmetric in $x_1, x_2, \ . \ . \ . \ x_n$. For the solution of these equations, we refer to Ex. 1, p. 38, and Ex. 3, p. 105, from which it will be readily inferred that ψ_1 can be expressed as a rational function of ϕ_1 in the following form:—

$$\nabla(\phi_1, \phi_2, \ . \ . \ . \ \phi_\rho)\ \psi_1 = A_0\phi_1{}^{\rho-1} + A_1\phi_1{}^{\rho-2} + \ . \ . \ . + A_{\rho-1},$$

where ∇ has the same meaning as in Art. 203, and $A_0, A_1, \ . \ . \ . \ A_{\rho-1}$ are all symmetric in $x_1, x_2, \ . \ . \ . \ x_n$.

It follows, conversely, that *two rational functions such that each can be expressed rationally in terms of the other belong to the* *same group*; for since each remains unchanged by all the substitutions which constitute the group of the other, it follows immediately that the two groups must coincide.

230. **Extension of Theorem with Corollaries.**—Even when the groups of ϕ_1 and ψ_1 are not identical, but one of them is included as a sub-group in the other, it is still true that the function which belongs to the wider group (and which has consequently the smaller number of distinct values) is expressible rationally in terms of the function of the narrower group.

Let ϕ_1 belong to the group G_1 of the preceding Article, and let ψ_1 belong to the wider group

$$G_1' = [1, \quad S_2, \quad . \ . \ . \ S_r, \quad S_{r+1}, \ . \ . \ . \ S_r'].$$

We have (Art. 226) the relations

$$r\rho = r'\rho' = N; \quad r' = kr; \quad \rho = k\rho';$$

there are, as before, ρ distinct values of ϕ; but the values of ψ, viz., $\psi_1, \psi_2, \psi_3, \ . \ . \ . \ \psi_\rho$, become equal in sets of k, so that only ρ' distinct values remain. It is still true, however, that the expression (1) of the preceding Article is a symmetric function of $x_1, x_2, \ . \ . \ . \ x_n$; for any substitution applied to it will reproduce in some order the distinct terms of the series. The equations (2) therefore can be solved as before, and an expression obtained for ψ_1 in terms of ϕ_1. In the final expression for ψ_1 in terms of ϕ_1, ϕ_1 may be replaced by ϕ_2 or ϕ_3 or etc. or ϕ_k, where these are the values of ϕ all associated with ψ_1, in such expressions as (1), and so derived from ϕ_1 by substitutions which do not alter ψ_1. If it be attempted, however, to express ϕ_1 in a similar form in terms of ψ_1, the solution fails, on account of the equality of two or more of the values of ψ; for it is implied in the solution of these equations that no two values of ϕ are equal (see Ex. 1, p. 38). What we can get by the equations is an expression for $\phi_1 + \phi_2 + \text{etc.} + \phi_k$ in terms of ψ_1, which would also follow immediately from Art. 229, as they both belong to the group of ψ_1. The theorem as extended was enunciated by Lagrange; it may be stated as follows:—

Lagrange's Theorem.—*If two rational functions of any set of variables are such that one remains unchanged by all the substitutions of the group to which the other belongs, the first is expressible by means of the second in the form of an integral polynomial whose coefficients are rational symmetric functions of the variables.*

From this proposition may be deduced important consequences which are contained in the following corollaries :—

COR. 1.—*A function can always be found in terms of which any number of given functions can be rationally expressed.*

The groups of the given functions have always one sub-group common to all, for the identical substitution $S = 1$ at least is common. Accordingly, the functions can all be expressed in terms of any one of the functions peculiar to the common sub-group. If ϕ, ψ, χ, . . . are the given functions, $\omega \equiv \alpha\phi + \beta\psi + \gamma\chi + \ldots$, where $\alpha, \beta, \gamma, \ldots$ are arbitrary constants, is one such function for the common sub-group; for any substitution which leaves it unaltered must leave $\phi, \psi, \chi, \ldots$ unaltered, and so must be common to the groups of $\phi_1, \psi_1, \chi_1, \ldots$

COR. 2.—*Any rational function whatever can be rationally expressed in terms of a function having N distinct values; in particular in terms of the Galois function.*

For the group of an N-valued function, reducing to the identical substitution, is included as a sub-group in every other.

COR. 3.—*The variables themselves can be expressed rationally in terms of the Galois function.*

The group to which x_1, for example, belongs contains $1 . 2 . 3 \ldots (n - 1)$ substitutions, including, of course, the sub-group unity. The n values of this function are the n variables $x_1, x_2, \ldots x_n$, and each can be expressed rationally in terms of the Galois function.

The proposition contained in this corollary was stated originally by Abel without proof. Galois has given a proof of the proposition founded on elementary principles, which we think it desirable to add, since it shows how the calculation may be conducted and the required rational expression for any one of the variables obtained.

Let $f(x) = 0$ be the equation whose roots are $x_1, x_2, \ldots x_n$, all supposed unequal; and let ψ_1 be a known value of a rational function ψ of the roots which has N distinct values.

If all the roots except x_1 be permuted in every possible way, we obtain $1 \cdot 2 \cdot 3 \ldots (n-1) = \mu$ distinct values of ψ given by the equation

$$F(\psi) = (\psi - \psi_1)(\psi - \psi_2) \ldots (\psi - \psi_\mu) = 0.$$

The coefficients of this equation when expanded are symmetric functions of $x_2, x_3, \ldots x_n$, and can therefore be expressed in the terms of the coefficients of

$$\frac{f(x)}{x - x_1} = 0,$$

and will involve x_1 in a rational form along with the coefficients of $f(x)$. If the expanded equation be represented by $F(\psi, x_1) = 0$, we have $F(\psi_1, x_1) = 0$, since it is satisfied by $\psi = \psi_1$; we have also $f(x_1) = 0$, from which it follows that the equations $f(x) = 0$ and $F(\psi_1, x) = 0$ have a common root. It is easily seen that this is the only root common. If therefore we seek the common measure of $f(x)$ and $F(\psi_1, x)$, as all the remainders vanish for $x = x_1$, in particular the remainder of the first degree in x gives for x_1 a rational expression in terms of ψ_1 and the coefficients of $f(x)$, in which expression ψ_1 may be replaced by ψ_2 or ψ_3 or etc. or ψ_μ without altering its value.

Ex. For a cubic equation

$$f(x) \equiv x^3 + p_1x^2 + p_2x + p_3 = 0,$$

if ψ be taken equal to the Galois function $a_1x_1 + a_2x_2 + a_3x_3$, it is readily proved that $F(\psi_1, x_1)$ involves x_1 in the second power, and the problem is reduced to finding the greatest common measure of a quadratic and cubic. The question is simplified by taking the special Galois function $x_1 + \omega x_2 + \omega^2 x_3 = \psi_1$; we find in this case that the coefficient of x_1^2 vanishes, and x_1 is obtained immediately in terms of ψ_1 as follows:—

$$x_1 = \frac{\psi_1^2 - p_1\psi_1 + p_1^2 - 3p_2}{3\psi_1}.$$

Cor. 4.—*All the values of the Galois function can be expressed rationally in terms of any one among them.*

For all belong to the same group unity.

231. **Two-valued Functions. Theorem.**—*Every two-valued integral function of n variables is of the form $S_1 \pm S_2\sqrt{\Delta}$, where S_1 and S_2 are integral symmetric functions, and Δ the discriminant.*

A two-valued function must belong to a group of order $\frac{1}{2}N$. The only group of this order is the alternate group (Ex. 5, Art. 226), to which the function $\sqrt{\Delta}$ belongs. The theorem therefore follows as an immediate consequence of the fundamental theorem of Art. 229. On account of its importance, however, we give the following independent proof:—

Let the two values of the function be denoted by ϕ_1 and ϕ_2, and let G_1 and G_2 be the corresponding groups, each of order $\frac{1}{2}N$. In the first place, these two groups must be identical; for if any substitution S of G_1 were to change ϕ_2 to its second value ϕ_1, then S^{-1} would change ϕ_1 into ϕ_2; but this is impossible, since S^{-1} as well as S belongs to the group G_1. Every substitution, therefore, of G_1 must belong to G_2, and *vice versa*.

To show now that these groups coincide with the alternate group, consider the function $\phi_1 - \phi_2 = \psi$. Any substitution which belongs to the common group leaves this unaltered; any other will change ϕ_1 to ϕ_2 and ϕ_2 to ϕ_1, and will therefore change the sign of ψ; some *transposition*, $(x_\alpha x_\beta)$ for example, will have this effect, for no group can include all transpositions without coinciding with the symmetric group. It is easily inferred that $\phi_1 - \phi_2$ is divisible by $x_\alpha - x_\beta$, and hence by the product of all the differences, since ψ^2 is symmetric.

The quotient of ψ by $\sqrt{\Delta}$ is symmetric. To prove this, let $(\sqrt{\Delta})^m$ be the highest power of $\sqrt{\Delta}$ which occurs in ψ. The quotient of ψ by $(\sqrt{\Delta})^m$ is symmetric, since, if not, it would be an alternating function, and would again contain $\sqrt{\Delta}$ as a factor, which is contrary to hypothesis. It follows immediately that

m is an odd number, and that the quotient of ϕ by $\sqrt{\Delta}$ is symmetric. Writing therefore $\phi_1 - \phi_2 = A\sqrt{\Delta}$, and $\phi_1 + \phi_2 = B$, where A and B are both symmetric, we at once derive

$$\phi_1 = S_1 + S_2\sqrt{\Delta}, \quad \phi_2 = S_1 - S_2\sqrt{\Delta},$$

where S_1 and S_2 are both symmetric functions of the variables $x_1, x_2, \ldots x_n$. It is, of course, also evident that the groups G_1 and G_2 coincide with the group of $\sqrt{\Delta}$, viz. the alternate group.

232. **Theorem.**—*The alternating functions are the only unsymmetric functions of n variables of which a power can be symmetric.*

The theorems contained in this and the next following Articles are of great importance in connexion with the problem of the general solution of algebraical equations. It will be sufficient to prove the theorem for prime powers; for if there exists a function $F(x_1, x_2, \ldots x_n)$ such that $F^{p.q}$ is symmetric, p being prime, then there is also a function $\phi \equiv F^q$ such that ϕ^p is symmetric. Let therefore

$$\phi^p \equiv S, \text{ a symmetric function.}$$

Since the group of ϕ, which is unsymmetric, cannot contain all the transpositions, let $\sigma \equiv (x_\alpha x_\beta)$ be a transposition which converts ϕ into ϕ_j; we have

$$\phi_j^p = \phi^p = S,$$

and therefore $\phi_j = \omega\phi$, where ω is a p^{th} root of unity. Hence

$$\sigma\phi = \phi_j = \omega\phi,$$

and, operating again with σ,

$$\sigma^2\phi = \omega\sigma\phi = \omega^2\phi;$$

but $\sigma^2 = 1$; hence $\omega^2 = 1$, and consequently $p = 2$.

Since therefore ϕ^2 is symmetric, ϕ is an alternating function, and the proposition is proved.

233. **Theorem.**—*For any number, n, of independent elements there is no multiple-valued function of which a power is two-valued when $n > 4$; and when $n = 3$, or $n = 4$, if there is any such power, it is a third power.*

Confining our attention as before to prime numbers, and supposing that ϕ is a multiple-valued function whose p^{th} power is two-valued, we have (Art. 231)

$$\phi^p = S_1 + S_2\sqrt{\Delta}. \tag{1}$$

The group of ϕ cannot contain all the circular substitutions of the third order, for if it did this group would coincide with the alternate group, and ϕ would be two-valued (Ex. 7, Art. 226). Let $\sigma \equiv (x_\alpha x_\beta x_\gamma)$ be such a substitution not contained in the group of ϕ, and suppose $\sigma\phi = \phi_j$. From the equation (1), since $S_1 + S_2\sqrt{\Delta}$ is unaltered by σ, we have

$$\phi^p = \phi_j{}^p;$$

hence $\phi_j = \omega\phi$, where ω is a p^{th} root of unity. Operating again twice in succession with σ, we obtain readily

$$\sigma\phi = \omega\phi,$$
$$\sigma^2\phi = \omega\sigma\phi = \omega^2\phi,$$
$$\sigma^3\phi = \omega^2\sigma\phi = \omega^3\phi;$$

whence, since $\sigma^3 = 1$, we have $\omega^3 = 1$, and therefore $p = 3$.

Again, when the number of elements is greater than 4, there are circular substitutions of the fifth order, and these cannot be all contained in the group of ϕ (Ex. 8, Art. 226). Let τ be one of those not contained in this group, and $\tau\phi = \phi_j$. We have, as before, from the equation (1), by applying this substitution (which does not affect the right-hand side),

$$\phi^p = \phi_j{}^p = S_1 + S_2\sqrt{\Delta}.$$

Hence, proceeding as before, we have $\tau\phi = \omega\phi$; and operating again on this and the succeeding equations with τ, we readily find $\tau^5\phi = \omega^5\phi$; whence $\omega^5 = 1$, since $\tau^5 = 1$, and it is proved that $p = 5$. Now, this result being inconsistent with the value of p previously obtained, viz. 3, we infer that when the number of elements is greater than 4, it is impossible to find any multiple-valued function ϕ, a prime power of which will be two-valued.

That there are actually, when n is not greater than 4, multiple-valued functions, a third power of which is two-valued, will appear from the following applications to the cases where $n = 3$ and $n = 4$:—

1. To find a multiple-valued function of three elements whose third power is two-valued. We examine whether the problem admits of solution by means of the simplest linear function, viz.

$$\phi = \alpha x_1 + \beta x_2 + \gamma x_3 ;$$

that is, whether the constants α, β, γ can be determined so as to make ϕ fulfil the required conditions.

Taking $\sigma \equiv (x_1 x_2 x_3)$, and identifying $\sigma\phi$ with $\omega\phi$, where $\omega^3 = 1$, we have

$$\alpha x_2 + \beta x_3 + \gamma x_1 \equiv \omega (\alpha x_1 + \beta x_2 + \gamma x_3) ;$$

whence

$$\gamma = \omega\alpha, \quad \beta = \omega^2\alpha ;$$

and immediately

$$\phi \equiv \alpha (x_1 + \omega^2 x_2 + \omega x_3).$$

Taking $\alpha = 1$, we infer that a function of the type $x_1 + \omega^2 x_2 + \omega x_3$ satisfies the conditions of the problem. This function is six-valued, and its cube two-valued (compare Art. 59, vol. i.).

The student will easily prove, in a similar manner, that any function of the type

$$x_1{}^m + \omega x_2{}^m + \omega^2 x_3{}^m,$$

where m is any integer, will equally well supply a solution of the problem.

2. To investigate a similar function when $n = 4$. In this case it is clear that no linear function of the type $\alpha x_1 + \beta x_2 + \gamma x_3 + \delta x_4$ can, without making $\delta = 0$, fulfil the condition of being multiplied by a factor when operated on by the substitution $\sigma \equiv (x_1 x_2 x_3)$. We take therefore the function next in simplicity, viz. one of the type

$$\phi = \alpha x_1 x_2 + \beta x_2 x_3 + \gamma x_3 x_1 + x_4 (\alpha' x_1 + \beta' x_2 + \gamma' x_3).$$

The function obtained from this by the operation of σ is

$$\phi_j = \alpha x_2 x_3 + \beta x_3 x_1 + \gamma x_1 x_2 + x_4 (\alpha' x_2 + \beta' x_3 + \gamma' x_1).$$

Identifying ϕ_j with $\omega\phi$, and replacing β, γ, β', γ' by their values in terms of α, α', we have

$$\phi = \alpha (x_1 x_2 + \omega^2 x_2 x_3 + \omega x_3 x_1) + \alpha' (x_1 x_4 + \omega^2 x_2 x_4 + \omega x_3 x_4).$$

Operating again with a different substitution of the third order, say $\tau \equiv (x_1 x_2 x_4)$, and denoting $\tau\phi$ by ϕ_k, we have

$$\phi_k = a(x_2 x_1 + \omega^2 x_4 x_3 + \omega x_3 x_2) + a'(x_2 x_1 + \omega^2 x_4 x_1 + \omega x_3 x_1).$$

Identifying, as before, ϕ_k with $\theta\phi$, where θ is some cube root of unity, we find at once $\theta = \omega^2$, and $a' = \omega^2 a$, the remaining relations being all consistent with these. We have therefore, taking $a = 1$,

$$\phi = x_1 x_2 + x_3 x_4 + \omega(x_1 x_3 + x_2 x_4) + \omega^2 (x_1 x_4 + x_2 x_3).$$

This is a function of the required kind, having itself six values, but only two values when cubed (compare Art. 66, vol. i., and Ex. 3, Art. 226).

Section III.—The Galois Resolvent.

234. **Galois Resolvent—Group of an Equation.**—Let

$$F(x) \equiv x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0 \qquad (1)$$

be an equation whose roots, supposed all unequal, are $x_1, x_2, \ldots x_n$, and whose coefficients are regarded as known rational quantities. If there are irrational quantities in the coefficients, they are associated with or adjoined to rational quantities, and all quantities obtained from the combination by addition, subtraction, multiplication or division are regarded in the following discussion as rational and called *rational.* They may also be described as being in the domain of the irrational quantities contained in the coefficients (see Art. 236). The Galois function

$$\psi_1 \equiv a_1 x_1 + a_2 x_2 + \ldots + a_n x_n$$

has N distinct values $\psi_1, \psi_2, \ldots \psi_N$, corresponding to the N substitutions of the symmetric group (Art. 227). The equation of the N^{th} degree whose roots are these N values, viz.,

$$\Psi(z) \equiv (z - \psi_1)(z - \psi_2) \ldots (z - \psi_N) = 0, \qquad (2)$$

is called the *Galois resolvent.* When this equation is expanded, the roots $x_1, x_2, \ldots x_n$ will enter it in a symmetric form; hence, the coefficients of z in the expanded equation can all be expressed *rationally* in terms of $p_1, p_2, \ldots p_n$. In general, this equation is irreducible; that is to say, it cannot be broken up into factors of inferior degree with *rational* coefficients. We proceed to examine under what circumstances it may become reducible. For this purpose, suppose $\Psi_1(z)$ to be an irreducible factor of the r^{th} degree, with *rational* coefficients, contained in $\Psi(z)$, and let

$$\Psi_1(z) = (z - \psi_1)(z - \psi_2) \ldots (z - \psi_r), \qquad (3)$$

where $\psi_2, \psi_3, \ldots \psi_r$ are derived from ψ_1 by means of the substitutions $S_2, S_3, \ldots S_r$. The following propositions can be established in reference to these substitutions :—

(1). *Every function* ϕ *of the roots which is unchanged by the substitutions* $1, S_2, S_3, \ldots S_r$ *can be expressed rationally in terms of* $p_1, p_2, \ldots p_n$.

By Art. 230, Cor. 2, ϕ may be expressed as a *rational* function of ψ_1 and the coefficients, say $f(\psi_1)$. Now, under the operation of the substitutions S_2, S_3 . . . ϕ remains unchanged, but ψ_1 becomes in succession $\psi_2, \psi_3, \ldots$; hence

$$f(\psi_1) = f(\psi_2) = f(\psi_3) = \ldots = \frac{1}{r}\{f(\psi_1) + f(\psi_2) + \ldots + f(\psi_r)\};$$

but the latter expression being symmetric in the roots of $\Psi_1 = 0$ can be *rationally* expressed by the coefficients of this equation, which are themselves *rational*.

(2). *Every function which is rationally expressible will be unchanged by the substitutions* $1, S_2, S_3, \ldots S_r$.

Let ϕ be a function of the roots which has a *rational* expression, say R; and let $f(\psi_1)$ be the function of ψ_1 by which ϕ can be also expressed (Art. 230). We have, then, $f(\psi_1) = R$; whence the equation $f(z) - R = 0$ has a root ψ_1 in common with the equation $\Psi_1(z) = 0$; but the latter equation is irreducible, and therefore all its roots must be common to the two equations (otherwise by finding the common measure of $\Psi_1(z)$ and $f(z) - R$, we would get a *rational* factor of $\Psi_1(z)$), and consequently $f(\psi_1)$ is unaltered when ψ_1 is replaced by

$\psi_2, \psi_3, \ldots$; that is to say, ϕ is unaltered by the substitutions which change ψ_1 into $\psi_2, \psi_3, \ldots \psi_r$, in succession.

(3). *The substitutions* $1, S_2, S_3, \ldots S_r$ *form a group.*

The effect of the operation of any one of these substitutions, say S_α, on $\Psi_1(z)$, is to leave the function unchanged, since its coefficients are *rational*, and therefore by (2) unaltered by S_α; the new values, therefore, of $\psi_1, \psi_2, \ldots \psi_r$ derived by this substitution must be identical with the first values, the order only differing; the effect of a second of the given substitutions, say S_β, is to reproduce in some order the same values of ψ. It follows that $S_\alpha S_\beta \psi_1 = \psi_\gamma = S_\gamma \psi_1$, and $\therefore S_\alpha S_\beta = S_\gamma$ as ψ_1 is an N-valued function; and the proposition is therefore proved.

The group developed above is called the *group of the equation.* This group is unique, for if $\Psi_2(z)$ were another irreducible *rational* factor of $\Psi(z)$, the group associated with it would leave $\Psi_1(z)$ unaltered, as $\Psi_1(z)$ is *rationally* expressible, and so each of its substitutions would be contained in the group of $\Psi_1(z)$; similarly, each substitution of the group of $\Psi_1(z)$ would be contained in the group of $\Psi_2(z)$, and hence the groups must be the same, and therefore also the degrees of $\Psi_1(z)$ and $\Psi_2(z)$ are equal. Furthermore, $\Psi(z)$ divided by $\Psi_1(z)$ is *rational*, and if irreducible its degree must be r, the same as $\Psi_1(z)$. If its degree is greater than r, it must be reducible, and must have an irreducible factor of degree r. Proceeding in this way, we see that $\Psi(z)$ is composed of irreducible factors of degree r, which all have the same group associated with them. By associating with rational quantities other irrational ones in addition to those possibly involved in the coefficients, we may possibly break up $\Psi_1(z)$ into factors of the same degree regarded as rational, and their common group must be a sub-group of the original group of the equation, since they would not alter $\Psi_1(z)$. The reasoning will apply also if any N-valued function of the roots is taken instead of the Galois function. In fact if Ψ_1, Φ_1 are *rational* irreducible factors of the equations for two N-valued functions ψ_1, ϕ_1, as the coefficients of Ψ_1 are *rational*, Ψ_1 is unaltered by the group of Φ_1, and similarly Φ_1 is unaltered by the group of Ψ_1, and therefore the groups coincide, and the degrees of the factors are equal. Furthermore, if T is a substitution not included in the group of Ψ_1, the coefficients of the equation whose roots are $T\psi_1$, $TS_2\psi_1$, $TS_3\psi_1$, . . . $TS_r\psi$ are *rational* as they are unaltered by the substitutions of the group; for if S_α changes ψ_β to ψ_γ it may be written $S_\beta^{-1}S_\gamma$ and so alters $TS_\beta\psi_1$ to $TS_\gamma\psi_1$, and hence S_α effects the same change in the arrangement of $T\psi_1$, $TS_2\psi_1$, . . . $TS_r\psi_1$ in any function of $T\psi_1$, $TS_2\psi_1$, . . . $TS_r\psi_1$ as it effects in the arrangement of ψ_1, ψ_2, . . . ψ_r in the same function of ψ_1, ψ_2, . . . ψ_r. The N values of ψ can thus be arranged in N/r sets, such that any symmetric function of the values in any set is unaltered by the substitutions of a group of substitutions of order r. Whether resolvable or not, the N factors of the Galois resolvent can thus be arranged in N/r factors of degree r having each the same group of order r. This arrangement of the N values ψ_1, ψ_2, . . . ψ_N of any N-valued function of x_1, x_2, . . . x_n corresponds to an arrangement of the N substitutions of the symmetric groups into N/r sets in a manner similar to that in Art. 226, but instead of multiplying the members $S_1 \equiv 1$, S_2, . . . S_r of the group G_1 of order r by Σ, we multiply Σ by S_1, S_2, . . . S_r. Associated with S_1, S_2, . . . S_r is a set $S_1\psi_1$, $S_2\psi_1$, . . . $S_r\psi_1$ of r values of ψ such that any symmetric function of them is unaltered by the substitutions of G_1. Associated with the set ΣS_1, ΣS_2, . . . ΣS_r is a set of r different values of ψ, viz. $\Sigma S_1\psi_1$, $\Sigma S_2\psi_1$, . . . $\Sigma S_r\psi_1$, such that any symmetric function of them is also unaltered by the substitutions of G_1. It is to be most carefully observed in this and other discussions that in this book the order of a product of substitutions is *left to right*, and not *right to left*, which in many ways would be more preferable. The group of an equation may be any sub-group of the symmetric, according to the special character of the given equation. The number of such sub-groups, however, among which the group of the equation is to be sought, is limited by the following proposition :—

The group of an irreducible equation is transitive.

A group is said to be *transitive* when it contains one or more substitutions whose effect is to replace any element whatever by another arbitrarily chosen. A transitive group, therefore, has in it substitutions which affect all the elements. Now let the group G of the equation be, if possible, not transitive, and let it affect only the elements $x_1, x_2, \ldots x_m (m < n)$. The substitutions of G, altering only among themselves the positions of these m roots, will leave their symmetric functions unaltered. These symmetric functions, therefore, are *rationally* expressible, and the function $F(x)$ will admit a *rational* divisor,

$$(x - x_1)(x - x_2) \ldots (x - x_m),$$

and will become reducible contrary to hypothesis.

EXAMPLES.

1. To form the sextic equation whose roots are the six values of the Galois function

$$a_1x_1 + a_2x_2 + a_3x_3,$$

and to express its coefficients in terms of the coefficients of the cubics $(a, b, c, d)(x, 1)^3$ and $(a'b'c'd')(x, 1)^3$, whose roots are x_1, x_2, x_3, and a_1, a_2, a_3 respectively.

The roots x_1, x_2, x_3 of $(a, b, c, d)(x, 1)^3 = 0$ may be expressed in the form

$$ax_1 + b = p + q, \; ax_2 + b = \omega p + \omega^2 q, \; ax_3 + b = \omega^2 p + \omega q,$$

where

$$pq = -H, \; p^3 + q^3 = -G. \quad \therefore 2p^3 = -G \pm \sqrt{G^2 + 4H^3} = -G \pm a\sqrt{\Delta},$$
$$2q^3 = -G \mp a\sqrt{\Delta}.$$

Expressing a_1, a_2, a_3 similarly, we obtain

$$3p = a(x_1 + \omega^2 x_2 + \omega x_3), \; 3q = a(x_1 + \omega x_2 + \omega^2 x_3), \; 3p' = a'(a_1 + \omega^2 a_2 + \omega a_3),$$
$$3q' = a'(a_1 + \omega a_2 + \omega^2 a_3).$$

Hence

$$9pq' = aa'(\psi_1 + \omega\psi_2 + \omega^2\psi_3), \; 9p'q = aa'(\psi_1 + \omega^2\psi_2 + \omega\psi_3),$$

where

$$\psi_1 = a_1x_1 + a_2x_2 + a_3x_3, \quad \psi_2 = a_1x_3 + a_2x_1 + a_3x_2 = (132)\,\psi_1,$$
$$\psi_3 = a_1x_2 + a_2x_3 + a_3x_1 = (123)\,\psi_1.$$

Hence

$$3(pq' + p'q + bb') = aa'\psi_1, \quad 3(\omega^2 pq' + \omega p'q + bb') = \psi_2,$$
$$3(\omega pq' + \omega^2 p'q + bb') = \psi_3.$$

Putting

$$aa'\psi_1 - 3bb' = 3z = 3(pq' + p'q), \quad \therefore z^3 = p^3q'^3 + p'^3q^3 + 3pqp'q'(pq' + p'q)$$
$$= \tfrac{1}{2}(GG' \pm aa'\sqrt{\Delta\Delta'}) + 3HH'z.$$

Hence ψ_1 satisfies $z^3 - 3HH'z - \frac{1}{2}(GG' \pm aa'\sqrt{\Delta\Delta'}) = 0$, and the process used shows that it is also satisfied by ψ_2, ψ_3. Hence if $aa'y - 3bb' = 3z$,

$$a^3a'^3(y - \psi_1)(y - \psi_2)(y - \psi_3) \equiv 27\{z^3 - 3HH'z - \tfrac{1}{2}(GG' \pm aa'\sqrt{\Delta\Delta'})\}.$$

If therefore $\Delta\Delta'$ is a perfect square, the equation in z is rational and the Galois resolvent has a rational factor, whose group is the alternate group, viz. 1, (132), (123).

The other factor is found by finding pp', qq' and obtaining as above

$$3(pp' + qq' + bb') = aa'\psi_1', \quad 3(\omega^2 pp' + \omega qq' + bb') = aa'\psi_2',$$
$$3(\omega pp' + \omega^2 qq' + bb') = aa'\psi_3',$$

where $\psi_1' = (23)\,\psi_1$, $\psi_2' = (31)\,\psi_1$, $\psi_3' = (12)\,\psi_1$, and therefore putting $aa'\psi_1' - 3bb' = 3z$, we obtain as above, $z^3 - 3HHz - \frac{1}{2}(GG' \mp aa'\sqrt{\Delta\Delta'}) = 0$, which equation is also satisfied by ψ_2', ψ_3'.

Hence, putting $aa'y = 3(z + bb')$, the other factor is

$$a^3a'^3(y - \psi_1')(y - \psi_2')(y - \psi_3') \equiv 27\{z^3 - 3HHz' - \tfrac{1}{2}(GG' \mp aa'\sqrt{\Delta\Delta'})\}.$$

The product of the two factors is rational, and gives the Galois resolvent. Now

$$a^2\Delta = G^2 + 4H^3 = (p^3 - q^3)^2 = (p - q)^2(\omega p - \omega^2 q)^2(\omega^2 p - \omega q)^2$$
$$= -a^6\Pi(x_1 - x_2)^2/27,$$

and hence if $\Pi(x_1 - x_2)^2$ expressed in terms of the coefficients is a perfect square, the Galois resolvent has a rational factor. As α_1, α_2, α_3 are given, the similar value of $\sqrt{\Delta'}$ is rational.

As the alternate is the only transitive sub-group of the symmetric in the case of three elements, the above is the only class of irreducible equations of the third degree having a reducible Galois resolvent.

2. To form the equation of the 24th degree, whose roots are the several values of the Galois function $\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 + \alpha_4x_4$; and, secondly, to determine the conditions that it can be resolved into rational factors, expressed in terms of the coefficients of the quartics

$$(a, b, c, d, e)(x, 1)^4 \text{ and } (a_1, b_1, c_1, d_1, e_1)(x, 1)^4,$$

whose roots are x_1, x_2, x_3, x_4 and $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ respectively.

The roots x_1, x_2, x_3, x_4 may be expressed in the form

$$ax_1 + b = \sqrt{y_1} + \sqrt{y_2} + \sqrt{y_3}, \quad ax_2 + b = -\sqrt{y_1} + \sqrt{y_2} - \sqrt{y_3},$$
$$ax_3 + b = \sqrt{y_1} - \sqrt{y_2} - \sqrt{y_3}, \quad ax_4 + b = -\sqrt{y_1} - \sqrt{y_2} + \sqrt{y_3},$$

where $\sqrt{y_1}\sqrt{y_2}\sqrt{y_3} = -\frac{1}{2}G$, and y_1, y_2, y_3 are roots of

$$y^3 + 3Hy^2 + (3H^2 - \tfrac{1}{4}a^2I)\,y - \tfrac{1}{4}G^2 = 0. \qquad (1)$$

Expressing a_1, a_2, a_3, a_4 similarly in terms of $\sqrt{\beta_1}, \sqrt{\beta_2}, \sqrt{\beta_3}$, where $\beta_1, \beta_2, \beta_3$ are roots of a similar equation obtained by substituting dotted letter for undotted, and $\sqrt{\beta_1}\sqrt{\beta_2}\sqrt{\beta_3} = -\frac{1}{2}G'$, we obtain

$$4\sqrt{y_1} = a\,(x_1 - x_2 + x_3 - x_4), \quad 4\sqrt{y_2} = a\,(x_1 + x_2 - x_3 - x_4),$$
$$4\sqrt{y_3} = a\,(x_1 - x_2 - x_3 + x_4),$$
$$4\sqrt{\beta_1} = a'\,(a_1 - a_2 + a_3 - a_4), \quad 4\sqrt{\beta_2} = a'\,(a_1 + a_2 - a_3 - a_4),$$
$$4\sqrt{\beta_3} = a'\,(a_1 - a_2 - a_3 + a_4).$$

Hence

$$16\sqrt{y_1}\sqrt{\beta_1} = aa'\,(\phi_1 - \phi_2 + \phi_3 - \phi_4), \quad 16\sqrt{y_2}\sqrt{\beta_2} = aa'\,(\phi_1 + \phi_2 - \phi_3 - \phi_4),$$
$$16\sqrt{y_3}\sqrt{\beta_3} = aa'\,(\phi_1 - \phi_2 - \phi_3 + \phi_4), \quad 16\,bb' = aa'\,(\phi_1 + \phi_2 + \phi_3 + \phi_4),$$

where

$$\phi_1 = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4, \quad \phi_2 = (1, 2)\,(3, 4)\,\phi_1, \quad \phi_3 = (1, 3)\,(2, 4)\,\phi_1,$$
$$\phi_4 = (1, 4)\,(2, 3)\,\phi_1.$$

Hence $4\,(\sqrt{y_1}\sqrt{\beta_1} + \sqrt{y_2}\sqrt{\beta_2} + \sqrt{y_3}\sqrt{\beta_3} + bb') = aa'\phi_1$, with similar values for ϕ_2, ϕ_3, ϕ_4, the corresponding signs being $-, +, -, +\ :\ +, -, -, +\ :\ -, -, +, +$.

Putting $aa'\phi_1 = 4\,(bb' + z)$,

$$\therefore z = \sqrt{y_1}\sqrt{\beta_1} + \sqrt{y_2}\sqrt{\beta_2} + \sqrt{y_3}\sqrt{\beta_3},$$
$$z^2 = \Sigma y_1\beta_1 + 2\Sigma\sqrt{y_2}\sqrt{\beta_2}\sqrt{y_3}\sqrt{\beta_3},$$
$$\therefore (z^2 - \Sigma y_1\beta_1)^2 = 4\Sigma y_2\beta_2 y_3\beta_3 + 8\sqrt{y_1}\sqrt{y_2}\sqrt{y_3}\sqrt{\beta_1}\sqrt{\beta_2}\sqrt{\beta}\,.\,z.$$

Hence, if

$$\psi_1 = y_1\beta_1 + y_2\beta_2 + y_3\beta_3, \quad \text{and} \quad \chi_1 = y_1^2\beta_1^2 + y_2^2\beta_2^2 + y_3^2\beta_3^2,$$
$$z^4 - 2\psi_1 z^2 - 2GG'z - \psi_1^2 + 2\chi_1 = 0,$$

which by the process used is also satisfied by ϕ_2, ϕ_3, ϕ_4.

Now by the last example ψ_1 is a root of a cubic equation involving the irrational quantity

$$\Pi\,(y_1 - y_2) = \Pi\,(\sqrt{y_1} + \sqrt{y_2})\,(\sqrt{y_1} - \sqrt{y_2}) = \Pi\tfrac{1}{2}a\,(x_1 - x_4)\,\tfrac{1}{2}a\,(x_3 - x_2)$$
$$= a^6\Pi\,(x_1 - x_2)\,/\,64,$$

and is obtained by calculating H_1, G_1 for the equation (1), and H_1', G_1' for the similar equation with dotted letters; and so putting $\psi_1 = 3\,(HH' + w)$ the equation is

$$w^3 - \frac{1}{48}a^2a'^2II'w - \frac{a^3a'^3}{864}\{27\,JJ' \pm \sqrt{DD'}\} = 0, \qquad (2)$$

where

$$a^{12}\Pi\,(x_1 - x_2)^2\,/\,64^2 = \Pi\,(y_1 - y_2)^2 = -27\,(G_1^2 + 4H_1^3)$$
$$= a^6\,(I^3 - 27J^2)\,/\,16 = a^6D\,/\,16,$$

and $\therefore$ D is the discriminant of the original quartic. The equation therefore involves the irrational quantity $\sqrt{D}$ only, as $\sqrt{D'}$ is rational for a_1, a_2, a_3, a_4 are given.

Now $\chi_1 = y_1{}^2\beta_1{}^2 + y_2{}^2\beta_2{}^2 + y_3{}^2\beta_3{}^2$ being a six-valued function of y_1, y_2, y_3 is by Art. 229 equal to a rational integral function of ψ_1, whose degree is 5, and which may be reduced to one of degree 2 by means of the cubic (2) which $\psi_1 = 3(HH' + w)$ satisfies.
Hence, putting $aa'\phi = 4(bb' + z)$, $\phi_1, \phi_2, \phi_3, \phi_4$ are roots of

$$z^4 - 2\psi_1 z^2 - 2GG'z + P\psi_1{}^2 + Q\psi_1 + R = 0, \tag{3}$$

where P, Q, R involve linearly the irrational quantity $\sqrt{D}$.

Eliminating ψ_1 from (2) and (3), we obtain an equation of the 12th degree in z involving $\sqrt{D}$, which therefore, if D is a perfect square, provides a rational factor of the Galois resolvent.

As the other roots, ψ_2, ψ_3, of the cubic (2) for ψ_1 are obtained by the substitutions (132), (123) operating on ψ_1 considered as a function of y_1, y_2, y_3, and as the alterations of y_1 to y_2, y_1 to y_3, y_2 to y_3 in the expressions for $\sqrt{y_1}, \sqrt{y_2}, \sqrt{y_3}$ in terms of x_1, x_2, x_3, x_4 are equivalent to alterations of x_2 to x_3, x_3 to x_4, x_2 to x_4 respectively, the other values of $\phi_1, \phi_2, \phi_3, \phi_4$ associated with ψ_2, ψ_3 are obtained by operating with (234) and (324) on $\phi_1, \phi_2, \phi_3, \phi_4$, thus obtaining 12 values whose group is the alternate group. Similarly, the cubic for $\psi_1', \psi_2', \psi_3'$, obtained by changing the sign of $\sqrt{D}$ in (2), since $\psi_1', \psi_2', \psi_3'$ are derived from ψ_1 by the alterations of y_2 to y_3, y_3 to y_1, y_1 to y_2 respectively, is associated with 12 values of ϕ_1 obtained by operating with (24), (34), (23) on $\phi_1, \phi_2, \phi_3, \phi_4$. If $\sqrt{D}$ is adjoined to the rational domain of the coefficients the group of the quartic becomes the alternate group. If, further, a root of the equation for ψ is adjoined, the group becomes 1, (1, 2) (3, 4), (1, 3) (2, 4), (1, 4) (2, 3); and we note that as the values of ψ can be expressed rationally in terms of any one value ψ_1 (Art. 230, Cor. 2), the other *rational* factors of the Galois Resolvent are the five obtained by substituting $\psi_2, \psi_3, \psi_1', \psi_2', \psi_3'$ for ψ_1 in (3); and further we verify that the group of each of these factors is 1, (1, 2) (3, 4), (1, 3) (2, 4), (1, 4) (2, 3) as this group is unaltered when transformed by any substitution.

3. To determine under what conditions the Galois resolvent breaks up into factors in the case of the quintic.

To find these conditions we may use any 120-valued function instead of the Galois function $\psi = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5$, and in particular may use the form of ψ obtained by replacing a_r by a^r, a being an imaginary fifth root of unity.

The function ψ has 120 values; and when a^r is put in place of a_r, with the condition $a^5 = 1$, the Galois resolvent takes the form

$$(\psi^5 - \psi_1{}^5)(\psi^5 - \psi_2{}^5) \ldots (\psi^5 - \psi_{24}{}^5) = 0;$$

for, if ψ_r is a root, so also are $a\psi_r, a^2\psi_r, a^3\psi_r, a^4\psi_r$.

We now put $\psi^5 = \theta$, and from the values of θ select the following four:—

$$\theta_1 = (ax_1 + a^2x_2 + a^3x_3 + a^4x_4 + x_5)^5,$$
$$\theta_2 = (a^2x_1 + a^4x_2 + ax_3 + a^3x_4 + x_5)^5,$$
$$\theta_3 = (a^3x_1 + ax_2 + a^4x_3 + a^2x_4 + x_5)^5,$$
$$\theta_4 = (a^4x_1 + a^3x_2 + a^2x_3 + ax_4 + x_5)^5,$$

of which the last three are obtained by substituting in succession α^2, α^3, α^4 for α in θ_1, and reducing by the equation $\alpha^5 = 1$. It should be noticed that, since 5 is a prime number, if in the series α, α^2, α^3, α^4 we replace α by α^p, the same roots are reproduced in a different order.

From θ_1, θ_2, θ_3, θ_4 the 24 values of θ can be obtained, in six sets of four, by the six permutations of x_1, x_2, x_3; for x_4, having all the multipliers possible viz. α, α^2, α^3, α^4, need not be permuted. Every symmetric function of θ_1, θ_2, θ_3, θ_4 has six values obtained by the same permutations. The resolvent is therefore the product of six quartics of the type

$$\theta^4 + \phi\theta^3 + \rho\theta^2 + \sigma\theta + \tau = 0.$$

Again, since $\sum_1^6 \phi_r^{\lambda}\tau_r^{\mu}$ is the sum of all the values $\phi^{\lambda}\tau^{\mu}$ can assume, it is unchanged by any substitution, the order only being affected; it is therefore expressible by the coefficients of the quintic; whence, making $\mu = 1$, we find by Art. 229 that τ is a rational function of ϕ. The same is true for all the coefficients; therefore if one is known, all are known. Now, let $\mu = 0$, then $\sum_1^6 \phi_r^{\lambda}$ is known, and we can therefore form a sextic for determining ϕ; and by adjoining one root of this sextic the equation for ψ (and therefore all equations for 120-valued functions) has a *rational* factor of degree 20 whose group is *either* that formed by combining the group common to θ_1, θ_2, θ_3, θ_4, viz. 1, A, A^2, A^3, A^4 where $A = (54321)$, with the group 1, B, B^2, B^3, where $B = (1342)$, and B, B^3, B^2 transform θ_1 to θ_2, θ_1 to θ_3, and θ_1 to θ_4 respectively, *or* its transformation by one of the substitutions giving the five permutations of x_1, x_2, x_3.

Thus the solution of the quintic depends on the solution of a sextic, as Lagrange has pointed out. The analogous method was successful in solving the cubic, by reducing it to a quadratic in ψ^3. In the case of the septimic, a similar treatment of the Galois resolvent would reduce it to 120 sextics in ψ^7.

Section IV.—The Algebraic Solution of Equations.

235. **Application of the Theory of Substitutions to the Algebraic Solution of Equations.**—The problem of the solution of an algebraic equation may be stated as follows:—From the given values of the single-valued functions, p_1, p_2, . . . , viz. the coefficients of the equation, to find the value of an N-valued function, viz. a root of the Galois resolvent; for we have seen (Art. 230, *Cor.* 3) that each of the roots x_1, x_2, . . . can be expressed rationally in terms of any Galois function. Although the actual determination of the roots in terms of the

given coefficients is not facilitated by this mode of procedure, yet the statement of the problem in this form is important in reference to the question of the possibility of the solution of algebraic equations generally.

The known solutions of the cubic and biquadratic may from this point of view be presented briefly as follows :—

(1). In the case of the cubic equation

$$x^3 + p_1x^2 + p_2x + p_3 = 0,$$

we have to find a six-valued function of the form $\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3$ from the given single-valued functions p_1, p_2, p_3 by the extraction of roots. In the first place, all two-valued functions can be expressed rationally (Art. 229) in terms of the two-valued function

$$\sqrt{\Delta_3} = \pm(x_1 - x_2)(x_1 - x_3)(x_2 - x_3),$$

and therefore in terms of p_1, p_2, p_3, along with the square root of a known function of the coefficients (Art. 42, Vol. I.) Now we have found (Art. 233, Ex. 2) a six-valued function $x_1 + \omega x_2 + \omega^2 x_3 \equiv \psi_1$, whose cube is two-valued. ψ_1 itself therefore can be expressed by means of a cube root of a function of the coefficients in addition to the square root previously introduced (cf. Art. 59, Vol. I.). A six-valued function having been thus obtained, the solution of the equation is theoretically complete.

(2). In the case of the biquadratic equation

$$x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = 0,$$

we have to find a 24-valued function of the form

$$\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3 + \alpha_4x_4$$

from the single-valued functions p_1, p_2, p_3, p_4 by the extraction of roots.

As in the preceding case, any two-valued function can be expressed rationally in terms of p_1, p_2, p_3, p_4, along with the two-valued function $\sqrt{\Delta_4}$, and hence in terms of these coefficients along with the square root of a known function of the coefficients (Ex. 15, p. 126, Vol. I.). Now, referring to Art. 233, Ex. 2, we find the six-valued function

$$\phi \equiv x_1x_2 + x_3x_4 + \omega\,(x_1x_3 + x_2x_4) - \omega^2\,(x_1x_4 + x_2x_3),$$

whose third power is two-valued ; ϕ will be expressible therefore by the aid of a cube root of a known function of the coefficients. We have now to find a means of passing from this six-valued function to a 24-valued function. The group of ϕ is (Ex. 3, Art. 226),

$$H \equiv [1,\ (12)\,(34),\ (13)\,(24),\ (14)\,(23)],\quad (\rho = 6,\ r = 4),$$

and a second function belonging to the same group is

$$\theta^2 \equiv (x_1 + x_2 - x_3 - x_4)^2\,(x_1x_3 + x_2x_4)^2.$$

This function is rationally expressible in terms of ϕ ; and the value of θ therefore is obtained in terms of the coefficients by the aid of an additional square root. The group of θ is

$$[1,\quad (12)\,(34)],\qquad (\rho = 12,\ r = 2),$$

to which the following function also belongs :—

$$\psi^2 = \{a_1\,(x_1 - x_2) + a_3\,(x_3 - x_4)\}^2\,;$$

ψ^2 is expressible in terms of θ ; and finally ψ, which is a 24-valued function, is obtained by the aid of another square root.

The process illustrated in these two cases may be described as the successive reduction of the group of an equation by the adjunction of definite radicals to the *rational* domain of the coefficients. The symmetric group is in each case first reduced to the alternate by the addition to the known coefficients of the square root of the discriminant. The further reduction depends on the included sub-groups of the alternate, till finally the group unity to which the Galois function belongs is reached. If the solution of the quintic were attempted by this method, we could proceed no further with the reduction than the first step, since, as has been seen (Art. 233), there exists in this case no multiple-valued function of the roots of which a power is two-valued. It cannot, however, be inferred immediately from this that the algebraic solution of the quintic is impossible. Before making this inference it will be necessary to examine closely the algebraic character of the formula which is the possible expression of a root of an algebraic equation; and hence to show the propriety of the application of the theory of substitutions to the problem.

For this purpose we proceed in the first place to explain the distinction between quantities which are to be regarded as rational and those which are to be regarded as irrational; or, in Kronecker's language, to define the *rational domain*.

236. **Definition of Rational Domain.**—All quantities obtained from certain parameters R', R'', R''' . . . combined with integers, by the operations of addition, subtraction, multiplication and division (including, therefore, raising to integral powers), constitute the rational domain $(R', R'', R''' \ldots)$ of $R', R'', R''' \ldots$.

The extraction of roots will, in general, lead to quantities outside the domain. We may, however, limit ourselves to the extraction of roots of prime order, since an $(mn)^{th}$ root can be replaced by an m^{th} root of an n^{th} root, and all numbers can be resolved into prime factors.

If the student refers to the expressions given for the roots of the quadratic, cubic, and biquadratic equations in terms of their coefficients, it will be found, when the roots are substituted in place of the coefficients, these expressions become rational functions of the roots involving the cube roots of unity, the rational domain consisting of the roots of the equations and the cube roots of unity.

It will appear subsequently if any algebraic formula which is an expression for a root of an equation of a higher degree exists, it must become a rational function of the roots (when they replace the coefficients) involving several primitive roots of unity; and finally, the theory of substitutions proves that functions of the roots do not exist satisfying such conditions, and that the algebraic solution of the higher equations is impossible.

237. **Form of the Roots of Equations algebraically solvable.**—If $f(x) = 0$ be an equation, the coefficients of which are included in the rational domain $(R', R'', R''', \ldots)$, we say that this equation is solvable algebraically when it is possible to satisfy the equation by substituting for x an expression formed of elements within the domain $(R', R'', \ldots)$ by means of the following operations of algebra, viz. addition, subtraction, multiplication, division (including therefore raising to integer powers), and the extraction of integer roots, the number of such operations being finite.

The value of x thus determined is designated as an *algebraic* function of the domain $(R', R'', R''', \ldots)$.

The building up of this algebraic function may always be completed in the following manner :—

1°. Form a rational function of the elements of the domain, viz.

$$F_\nu(R', R'', R''' \ \ldots).$$

2°. Let V_ν be definitely one of the p_ν quantities satisfying the equation

$$V_\nu^{p_\nu} = F_\nu(R', R'', R''' \ \ldots),$$

where p_ν is a prime number. We also suppose that F_ν is not an exact $p_\nu{}^{th}$ power, for if it was, V_ν would be included in the primitive domain.

3°. Adjoining V_ν to the primitive domain, form a rational function $F_{\nu-1}(V_\nu, R', R'', R''' \ldots)$ in this extended domain, and let $V_{\nu-1}$ be definitely one of the $p_{\nu-1}$ quantities satisfying the equation

$$V_{\nu-1}^{p_{\nu-1}} = F_{\nu-1}(V_\nu, R', R'', R''' \ \ldots),$$

where $p_{\nu-1}$ is a prime number. We also suppose that $F_{\nu-1}$ is not an exact $p_{\nu-1}{}^{th}$ power, for if so, $V_{\nu-1}$ would be included in the domain $(V_\nu, R', R'', \ldots)$.

4°. Adjoining $V_{\nu-1}$ to the last domain, form a new rational function in this new domain $F_{\nu-2}(V_{\nu-1}, V_\nu, R', R'', \ldots)$, and so on.

We can therefore represent the formation of the algebraic function x_1 where $f(x_1) = 0$, by the following chain of equations :—

$$\left.\begin{aligned}
V_\nu^{p_\nu} &= F_\nu (R', R'', \ldots),\\
V_{\nu-1}^{p_{\nu-1}} &= F_{\nu-1} (V_\nu, R', R'', \ldots),\\
V_{\nu-2}^{p_{\nu-2}} &= F_{\nu-2} (V_{\nu-1}, V_\nu, R', R'', \ldots),\\
&\cdots\\
V_1^{p_1} &= F_1 (V_2, V_3, \ldots V_\nu, R', R'', \ldots)\\
x_1 &= F_0 (V_1, V_2, \ldots V_\nu, R', R'', \ldots),
\end{aligned}\right\} \quad (A)$$

where the functions F are rational and the numbers p prime.

Before proceeding further, it is necessary to express the functions F in an integral form, if they are not so expressed already; and to fix our ideas we shall take $\nu = 3$, the method being the same in every case. Supposing F_1 not an integral function of V_2 and V_3, we can always put

$$F_1 = \frac{\phi (V_2, V_3)}{\psi (V_2, V_3)},$$

ϕ and ψ being *rational* and integral functions.

From the chain of equations we have in this case

$$V_3^{p_3} = F_3 (R), \quad V_2^{p_2} = F_2 (V_3, R), \quad R \equiv (R', R'', \ldots).$$

Also, if ω be a primitive root of the equation $x^{p_2} - 1 = 0$,

$$\psi(V_2, V_3)\, \psi (\omega V_2, V_3)\, \psi (\omega^2 V_2, V_3) \ldots \psi (\omega^{p_2 - 1} V_2, V_3) = \Psi_1(V_2^{p_2}, V_3).$$

Again, the product of these factors, omitting the first, is in the *rational* domain of V_2, V_3, as when expanded it does not contain ω. Now, eliminating $V_2^{p_2}$ by means of the equation

$V_2^{p_2} = F_2 (V_3, R)$, $\quad \Psi_1 (V_2^{p_2}, V_3)$ becomes $\Psi_2 (V_3, R)$.

Treating Ψ_2 in a similar manner, it is converted into a function of the form $\Psi_3 (V_3^{p_3}, R)$, the multiplier being in the *rational* domain of V_3; now eliminating $V_3^{p_3}$, Ψ_3 becomes $\Psi_4 (R)$.

Finally, multiplying the numerator ϕ by these factors, which were applied to ψ, &c., &c., the value of F_1 is unaltered and the denominator is a function of $R \equiv (R', R'', R''' \ldots)$. Thus F_1 is expressed as a *rational* function of V_2, V_3, in an integral form.

And therefore, in general, we are enabled to write *any rational* function of the V's, viz. F_{a-1}, as follows :—

$$F_{a-1}(V_a, V_{a+1} \ldots V_\nu, R) = J_0 + J_1 V_a + J_2 V_a^2 \ldots + J_{p_a - 1} V_a^{p_a - 1},$$

where the functions J are integral functions of V_{a+1}, V_{a+2}, $\ldots V_\nu$, and fractional only in R', R'', ...

It will be necessary now to prove a fundamental theorem of Abel's of which use will subsequently be made.

238. **Theorem.**—*If the equations*

$$f_1 x^{p-1} + f_2 x^{p-2} + \ldots + f_p = 0, \qquad (1)$$

$$x^p - F = 0, \qquad (2)$$

where p is a prime number, are simultaneously satisfied, either $f_1, f_2, f_3, \ldots f_p$ all vanish, or else one of the roots of the equation (2) *can be expressed rationally in terms of $f_1, f_2, \ldots f_p$ and F.*

For, suppose the coefficients of equation (1) not to vanish, then the equations (1) and (2) have a greatest common divisor

$$x^\rho + g_1 x^{\rho-1} + g_2 x^{\rho-2} + \ldots + g_\rho = 0, \qquad (3)$$

the coefficients of which are rational functions of $F, f_1, f_2, \ldots f_p$. Now if x_1 be any one of the roots common to the equations (1) and (2), the other roots will be of the form

$$\omega^a x_1, \omega^\beta x_1 \ \ldots \ \text{where } \omega^p - 1 = 0 ;$$

whence

$$g_\rho = x_1{}^\rho \omega^{a+\beta} \ \ldots = \omega^\pi x_1{}^\rho. \qquad (4)$$

Again, since p is a prime number, we can find two numbers m and n, of which one is negative, such that $mp + n\rho = 1$. Also

$$g_\rho^n = \omega^{n\pi} x_1^{n\rho} = \omega^{n\pi} x_1{}^{(1-mp)},$$

and therefore by (2)

$$\omega^{n\pi} x_1 = g_\rho^n F^m ;$$

therefore $\omega^{n\pi}x_1$, which is a root of equation (2), is expressed rationally in terms of $F, f_1, f_2, \ldots f_p$.

239. We proceed now to make a further reduction in the form of

$$F_{\alpha-1} = J_0 + J_1 V_\alpha + J_2 V_\alpha^2 + \ldots + J_{p_\alpha - 1} V^{p_\alpha - 1},$$

so that J_1 may be equal to unity.

Let J_κ be one of the coefficients $J_1, J_2 \ldots$ which does not vanish, and putting

$$J_\kappa V_\alpha^\kappa = W_\alpha,$$

there are integer numbers m and n, of which one is negative, such that

$$m\kappa + np_\alpha = 1;$$

whence

$$J_\kappa^m V_\alpha^{m\kappa} = J_\kappa^m V_\alpha^{(1-np_\alpha)} = W_\alpha^m;$$

therefore, we have

$$V_\alpha = W_\alpha^m F_\alpha^n J_\kappa^{-m}, \quad \cdot \cdot \quad F_\alpha = V_\alpha^{p_\alpha}.$$

Hence V_α and W_α can be expressed the one in terms of the other and the elements $V_{\alpha+1}, V_{\alpha+2}, \ldots V_\nu$, so that the rational domains $(V_\alpha, V_{\alpha+1}, \ldots V_\nu, R', R'', \ldots)$ and $(W_\alpha, V_{\alpha+1}, \ldots V_\nu, R', R'' \ldots)$ are equivalent.

Again, there is no power of W_α lower than p_α which is rational in this domain. For, if

$$W_\alpha^q = \Phi(V_{\alpha+1}, V_{\alpha+2}, \ldots V_\nu),$$

where $q < p_\alpha$,

$$J_\kappa^q V_\alpha^{\kappa q} = \Phi(V_{\alpha+1}, V_{\alpha+2}, \ldots V_\nu);$$

but κq is not divisible by p_α, for p_α being a prime number should divide κ or q; but both are less than p_α, and hence putting $\kappa q = mp_\alpha + r$ there is a power r of V_α less than p_α which is rationally expressible, but this is impossible, p_α being the lowest power of V_α which is a rational function of $V_{\alpha+1}, V_{\alpha+2}, \ldots V_\nu$.

Moreover, by raising W_α to the power p_α we have

$$W_\alpha^{p_\alpha} = J_\kappa^{p_\alpha} F_\alpha^\kappa = \Psi(V_{\alpha+1}, V_{\alpha+2}, \ldots V_\nu, R', R'' \ldots);$$

whence we learn that W_α, like V_α, is given by a binomial equation of the degree p_α, and we can replace the one by the other in the chain of equations connecting the V_α.

It follows that we can introduce W_α in place of V_α where it occurs in the functions $F_{\alpha-1}, F_{\alpha-2}, \ldots F_1$.

Therefore, in

$$F_{\alpha-1} = J_0 + J_1 V_\alpha + J_2 V_\alpha^2 + \ldots + J_{p_\alpha - 1} V_\alpha^{p_\alpha - 1},$$

when we replace $J_h V_\alpha^h$ by its value $J_h (F_\alpha^n J_\kappa^{-m})^h W_\alpha^{mh}$, this function is of the form $L_{h'} W_\alpha^{h'}$, where $mh = lp_\alpha + h'$, and $L_{h'}$ is a rational function of $V_{\alpha+1}, V_{\alpha+2}, \ldots V_\nu$, which can be rendered integral by Art. 237.

It should be noticed that when h is given the values 1, 2, 3, 4, . . . $p_\alpha - 1$ in the equation $mh = lp_\alpha + h'$, h' has for its values 1, 2, 3, . . . $p_\alpha - 1$ in some order, since all its values are distinct and less than p_α; also since $m\kappa + np_\alpha = 1$, κ is the only value of h for which the remainder $h' = 1$.

We see then that

$$F_{\alpha-1} = J_0 + W_\alpha + L_2 W_\alpha^2 + \ldots + L_{p_\alpha - 1} W_\alpha^{p_\alpha - 1},$$

where the L's have been rendered integral and $L_1 = 1$, and we return to the old notation by putting V_α for W_α, and $J_1 = 1$. We have then, finally, the important result

$$F_{\alpha-1}(V_\alpha, V_{\alpha+1} \ldots V_\nu, R) = J_0 + V_\alpha + J_2 V_\alpha^2 + \ldots + J_{p_\alpha - 1} V_\alpha^{p_\alpha - 1};$$

whence expanding the function

$$F_0 (V_1, V_2, \ldots V_\nu, R', R'' \ldots) = x_1,$$

a root of the equation of $f(x) = 0$, in powers of V_1 (the V with lowest suffix), and making the foregoing reductions, we have

$$x_1 = G_0 + V_1 + G_2 V_1^2 + \ldots + G_{p_1 - 1} V_1^{p_1 - 1}.$$

240. We proceed now to apply this theory to the solution of equations which are solvable algebraically.

For this purpose, forming the different powers of x_1, and taking care to reduce the exponents of $V_1, V_2 \ldots$ so as to be respectively less than $p_1, p_2, \ldots$ by means of the chain of equations which define $V_1, V_2 \ldots$ we shall arrive at the result

$$f(x_1) = H_0 + H_1V_1 + H_2V_1^2 + \ldots + H_{p_1-1}V_1^{p_1-1} = 0$$

by hypothesis, where $H_0, H_1, H_2 \ldots$ are integer functions of the V's.

By Abel's theorem $H_0, H_1, \ldots H_{p_1-1}$ must all vanish; for if not, the equations

$$H_0 + H_1V_1 + \ldots + H_{p_1-1}V_1^{p_1-1} = 0, \; V_1^{p_1} = F_1(V_2, V_3 \ldots V_\nu, R', R'', \ldots),$$

would be simultaneously satisfied, and F_1 would be an exact $p_1{}^{th}$ power in the domain $(V_2, V_3, \ldots V_\nu)$, which is contrary to hypothesis.

In a similar manner, expanding H_j in powers of V_2, viz.

$$H_j = K_0 + K_1V_2 + K_2V_2^2 \ldots + K_{p_2-1}V_2^{p_2-1},$$

the coefficients $K_0, K_1, \ldots$ should all vanish for exactly analogous reasons. But if V_2 be absent, expand in powers of V_3, &c., &c.

If in any case the coefficients of these successive functions do not vanish when arranged in powers of V_t, their indices having been reduced as much as possible, it is a proof that we have neglected to secure that each function F in the chain of equations is not an exact power, or that the number of the elements V has not been reduced to a minimum.

We have an example of this deficient reduction in the case of the cubic equation which we insert now, as an illustration.

Let $f(x) = x^3 + 3Px - 2Q$,

$$x_1 = \sqrt[3]{Q + \sqrt{Q^2 + P^3}} + \sqrt[3]{Q - \sqrt{Q^2 + P^3}}.$$ Vol. I., p. 45.

The chain of equations is as follows :—

$$V_3^2 = Q^2 + P^3, \quad V_2^3 = Q + V_3, \quad V_1^3 = Q - V_3, \qquad (A)$$

$$x_1 = V_1 + V_2;$$

whence $\frac{1}{3}f(x_1) = PV_2 + (V_2^2 + P)V_1 + V_2V_1^2$;

the coefficients of this equation, P, $V_2^2 + P$, V_2 cannot vanish identically, which is a proof that the functions V have not been reduced to a minimum number; and we proceed to show that V_1 is a part of the rational domain (V_2, V_3, P, Q).

From the chain of equations (A),

$$(V_1V_2)^3 = Q^2 - V_3^2 = -P^3; \text{ whence } V_1V_2 = -P.$$

Hence, $$V_1 = -\frac{P}{V_2} = -\frac{PV_2^2}{Q+V_3} = \frac{(Q-V_3)V_2^2}{P^2};$$

and so V_1 is a part of the domain (V_2, V_3, P, Q).

The chain of equations (A) is therefore reduced to

$$V_3^2 = Q^2 + P^3, \quad V_2^3 = Q + V_3, \quad x_1 = V_2 - \frac{P}{V_2} = V_2 + \frac{(Q-V_3)V_2^2}{P^2}.$$

The other two roots are obtained by putting ωV_2 and $\omega^2 V_2$ for V_2, the last element of the chain (A), and therefore are

$$x_2 = \omega V_2 + \frac{Q-V_3}{P^2}\omega^2 V_2^2,$$

$$x_3 = \omega^2 V_2 + \frac{Q-V_3}{P^2}\omega V_2^2.$$

Resuming the general investigation, we have

$$x_1 = G_0 + V_1 + G_2V_1^2 + G_3^3V_1^3, \ldots \tag{1}$$
$$f(x_1) = H_0 + H_1V_1 + H_2V_1^2 + H_3V_1^3 \ldots \equiv 0,$$

the coefficients H all vanishing.

Now, substituting in (1) for V_1, $\omega_1 V_1$, $\omega_1^2 V_1$, $\ldots$ $\omega_1^{p_1-1} V_1$, but leaving V_2, V_3, $\ldots$ V_ν fixed, we obtain the values of $x_2, x_3, \ldots x_{p_1}$, where $\omega_1^{p_1} - 1 = 0$, from the system of equations

$$x_{\kappa+1} = G_0 + \omega_1^\kappa V_1 + G_2\omega_1^{2\kappa} V_1^2 \ldots \quad (\kappa = 0, 1, 2, \ldots p_1 - 1);$$

and finally, from this system of equations we have

$$V_1 = \frac{1}{p_1}\sum \omega_1^{-\kappa} x_{\kappa+1}; \tag{2}$$

whence we conclude that the irrational function of the coefficients V_1 is a rational function of the roots when the primitive root of unity ω_1 is adjoined to the rational domain.

In a similar way by keeping fixed in (1) any set of values of $V_2, V_3, \ldots V_\nu$, and for V_1 substituting $\omega_1 V_1, \omega_1{}^2 V, \ldots \omega_1{}^{p_1-1} V_1$, we get all the values of V_1 expressed as linear functions of the roots rational in the domain formed by the roots and the primitive root ω_1. It is not necessarily to be supposed that (1) gives a different root for every combination of values of $V_1, V_2, \ldots V_\nu$. The roots may occur in cycles, the same cycle of n roots being obtained for every value of one or more of $V_1, V_2, \ldots V_\nu$. In the cubic we have V_1, V_2 with $p_1 = 3, p_2 = 2$, and the same three roots occur for $\pm V_2$. In the quartic we have V_1, V_2, V_3, V_4 with $p_1 = 2, p_2 = 2, p_3 = 3, p_4 = 2$, and the same cycle of four roots occur for every value of V_3 or V_4.

There are $p_1 p_2 \ldots p_\nu$ values of V_1, as thus calculated, but they may all occur in cycles, all the values being reproduced for all values of some one or more of $V_2, V_3, \ldots V_\nu$.

We thus derive that every one of the values of V_1 is expressible as a linear function of the roots. These values of V_1 form all the functions obtained from one value by permuting the roots in every way, for the product $\Pi(x - V_1)$ of all the values of $x - V_1$ got by giving V_1 every possible value, is equal to $\Pi(x^{p_1} - V_1^{p_1})$, since if V_1 is a value of V_1 so is also $\omega_1 V_1, \omega_1^2 V_1, \ldots \omega_1^{p_1-1} V_1$, and $\Pi(x^{p_1} - V_1^{p_1})$ is rational since it is for similar reasons independent of the values of $V_2, V_3, \ldots V_\nu$.

Now to see that V_2 would be similarly expressible as a homogeneous function of degree p_1 rational in the domain formed by the roots and the primitive roots ω_1, ω_2, such that $\omega_1^{p_1} = 1$, $\omega_2^{p_2} = 1$; in

$$V_1^{p_1} = L_0 + V_2 + L_2 V_2^2 \ldots + L_{p_2-1} V_2^{p_2-1}$$

keep any set of values of $V_3, V_4, \ldots V_\nu$ fixed, and substitute $V_2, \omega_2 V_2, \omega_2^2 V_2, \ldots \omega_2^{p_2-1} V_2$, and in each case the corresponding values, say, $y_1, y_2 \ldots y_{p_2}$ of $V_1^{p_1}$, as already obtained, we get $V_2 = \frac{1}{p_2} \sum \omega_2{}^{-\kappa+1} y_\kappa$. We thus see that all the $p_2 p_3 \ldots p_\nu$ values of V_2 are expressible as stated, and as we showed for V_1 we may also show for V_2 that all the values may be derived from one value by permuting the roots in every possible way. We now see as the equations (A) are all of type

$$V_a^{p_a} = J_0 + V_{a+1} + J_2 V_{a+1}^2 \dots + J_{p_{a+1}-1} V_{a+1}^{p_{a+1}-1}$$

that we may successively express $V_3, V_4, \dots V_\nu$ as homogeneous functions of degrees p_1p_2, $p_1p_2p_3$, &c., of the roots, rational in the domain formed by the roots and the primitive roots $\omega_1, \omega_2 \dots \omega_\nu$, and that the values of any one V_a are derived by permuting the roots in every possible way.

Summing up the results arrived at, we have the following :—

Theorem.—*If an equation $f(x) = 0$, the coefficients of which are rational functions of the quantities $R', R'', \dots$ can be satisfied by an explicit algebraic function*

$$x = F(V_1, V_2, \dots V_\nu, R', R'', \dots),$$

the quantities V are rational and integral functions of the roots, and of the primitive roots of unity ; they are, moreover, determined by a chain of equations of the form

$$V_a^{p_a} = F_a(V_{a-1}, V_{a-2} \dots V_\nu, R', R'' \dots),$$

wherein the indices p are all prime numbers, and the functions F all rational.

This theorem makes it possible to apply the theory of substitutions to the proof of the proposition that general equations of degree higher than the fourth are not algebraically solvable. The proof is as follows :—

It has been shown that the first irrational function V_ν is the p_ν^{th} root of a function rational in the domain $(R', R'' \dots)$, and as V_ν is a rational function of the roots such that V_ν^p is symmetrical, it is, by Art. 232, the square root of the discriminant Δ, or of the form $S\sqrt{\Delta}$, where S is a symmetric function of the roots. Consequently, $p_\nu = 2$.

If we adjoin $S\sqrt{\Delta}$ to the rational domain, we include all the one-valued and two-valued functions of the roots. Proceeding another step, there must be a rational function of the roots $V_{\nu-1}$, which is $2p_{\nu-1}$ valued, and of which the $p_{\nu-1}{}^{th}$ power is two-valued, but no such function exists when $n > 4$ (Art. 233). Consequently the process, which should have led to the roots, cannot be continued.

We conclude, therefore, that *the general equation of degree higher than the fourth cannot be solved algebraically.*

In the foregoing investigation we have followed closely the systematic treatment of this question given by Netto in his *Substitutionentheorie.* The principles on which the investigation rests are due to Abel, who was the first to establish in a rigorous manner the impossibility of the algebraic solution of equations of a degree higher than the fourth. The fundamental theorem of the present article was stated by him in the following form :— *If an algebraic equation is solvable algebraically, we can always give to the root such a form that all the algebraic functions of which it is composed can be expressed rationally in terms of the roots of the proposed equation* (Abel, *Œuvres Complètes*, 1881, Vol. I., p. 75). The manner in which this theorem is applied in the proof given above is a modification of Abel's proof introduced by Wantzel, to whom the propositions, in the theory of substitutions, of Arts. 232 and 233, appear to be due (see Serret's *Cours d'Algèbre Supérieure*, Vol. II., p. 484).

For further information relative to substitutions and groups the reader is referred to *The Theory of Groups*, by Professor W. Burnside, Cambridge, 1911, and *The Theory of Equations*, by Professor Cajori, New York, 1904.

We think it desirable to add a section on Abelian equations, as the Galois resolvent or any equation whose roots are the N values of any rational N-valued function of the roots $x_1, x_2, \ldots x_n$ of an equation $f(x) = 0$ can be seen in a variety of ways to be of such type, and hence their solution made to depend on the solution of equations of degrees lower than N, in addition, of course, to depending on the solution of $f(x) = 0$.

Section V.—Abelian Equations.

241. **Definition of Abelian Equations. The Galois Resolvent is Abelian.**—An Abelian equation is such that its roots may be arranged in m sets of p each, and that the roots $x_1, x_2, \ldots x_p$ of each set are related as follows :—

$$x_2 = \theta(x_1),\ x_3 = \theta(x_2) = \theta^2(x_1),\ x_4 = \theta(x_3) = \theta^3(x_1),\ \ldots$$
$$x_p = \theta(x_{p-1}) = \theta^{p-1}(x_1),\ x_1 = \theta(x_p) = \theta^p(x_1),$$

where $\theta(x)$ is a rational function of x.

For instance, the N roots of the Galois resolvent, or of any equation $F(\phi) = 0$ whose roots are the N values $\phi_1, \phi_2, \ldots \phi_N$ of any N-valued rational function of the n roots $x_1, x_2 \ldots x_n$ of an equation $f(x) = 0$, are so related. But in such a case the division into sets may be effected in a variety of ways. To prove this we note that if S is any substitution whatsoever, by Art. 229, $S\phi_1 = \theta(\phi_1)$, where θ is a rational and integral function of degree $N - 1$, which is the same for every pair of roots derived from ϕ_1 and $S\phi_1$ by any substitution T, so that $ST\phi_1 = \theta(T\phi_1)$. This last result follows also by regarding $S\phi_1 = \theta(\phi_1)$ as an identity involving $x_1, x_2 \ldots x_n$ only, obtained by substituting for the coefficients of $f(x)$ their symmetrical expressions in terms of the roots, and hence as $S\phi_1 = \theta(\phi_1)$, $ST\phi_1 = T\theta(\phi_1) = \theta(T\phi_1)$. Now some power of S equals unity, say $S^p = 1$, and accordingly arrange the roots in sets of p, each of the type

$$T\phi_1,\ ST\phi_1,\ S^2T\phi_1\ \ldots\ S^{p-1}T\phi_1,$$

where we take $T = 1$ for the first set, and for each subsequent set take for value of T a substitution which has not been used up to that stage. We proceed in this way until all substitutions have been used up, just as in Art. 226. Now, as $ST\phi_1 = \theta(T\phi_1)$, taking S^mT for T, we have $S^{m+1}T\phi_1 = \theta(S^mT\phi_1)$, hence along with $ST\phi_1 = \theta(T\phi_1)$ we have

$$S^2T\phi_1 = \theta(ST\phi_1) = \theta^2(T\phi_1),\ S^3T\phi_1 = \theta(S^2T\phi_1) = \theta^3(T\phi_1),$$

and so on, ending with $T\phi_1 = S^pT\phi_1 = \theta(S^{p-1}T\phi)$, and hence the equation $F(\phi) = 0$ is Abelian.

242. **Solution of a General Abelian Equation.**—The roots of an Abelian equation may be obtained by solving an equation of degree p whose coefficients are rational integral functions of a root of an equation of degree m. We shall prove this by taking a particular case with $p = 3$, $m = 4$, and it will be easily seen that the theorem is true generally.

Let $q_1 = q(x_1, x_2, x_3)$ be any rational symmetric function of the three roots in the first set, q_2 the same function of the roots x_4, x_5, x_6 in the second set, and so on. We have

$$q_1 = q(x_1, x_2, x_3) = q\{x_1, \theta(x_1), \theta^2(x_1)\} = \phi(x_1),$$

where ϕ is a rational function of (x_1). Also as q_1 is a symmetric function $q_1 = q(x_2, x_3, x_1) = q\{x_2, \theta(x_2), \theta^2(x_2)\} = \phi(x_2)$. Similarly, $q_1 = \phi(x_3)$ $\therefore q_1 = \frac{1}{3}\{\phi(x_1) + \phi(x_2) + \phi(x_3)\}$. Similarly, $q_2 = \frac{1}{3}\{\phi(x_3) + \phi(x_4) + \phi(x_5)\}$, and so with similar values for q_3, q_4, we have

$$\Sigma q_1 = q_1 + q_2 + q_3 + q_4 = \tfrac{1}{3}\{\phi(x_1) + \phi(x_2) + \ldots + \phi(x_{12})\},$$

and $\therefore$ Σq_1 is a rational function of the coefficients of the equation $f(x) = 0$, whose roots are $x_1, x_2, \ldots x_{12}$. In precisely the same way we prove that Σq_1^2, Σq_1^3, Σq_1^4 are rational functions of the coefficients of $f(x)$. Now expressing the coefficients of

$$(y - q_1)(y - q_2)(y - q_3)(y - q_4)$$

by Newton's formulæ in terms of sums of powers of q_1, q_2, q_3, q_4, we derive that q_1, q_2, q_3, q_4 are roots of an equation of the fourth degree whose coefficients are rational functions of the coefficients of $f(x)$.

Furthermore, if r_1, r_2, r_3, r_4 denote any other symmetric function of the roots in the four sets respectively, we see in precisely the same way that Σr_1, $\Sigma r_1 q_1$, $\Sigma r_1 q_1^2$ and $\Sigma r_1 q_1^3$ are rational functions of the coefficients of $f(x)$, and hence by Ex. 1, p. 38, r_1, r_2, r_3, r_4 are rational integral functions of q_1, q_2, q_3, q_4 respectively, which functions are the same for all four pairs.

Hence putting $q_1 = x_1 + x_2 + x_3 = y_1$, we see by giving r_1 the values $x_2x_3 + x_3x_1 + x_1x_2$ and $x_1x_2x_3$ that the latter are rational integral functions of y_1. Hence x_1, x_2, x_3 are roots of a cubic equation $x^3 - y_1x^2 + \phi(y_1)x - \psi(y_1) = 0$, where ϕ, ψ are rational integral functions, and we note this cubic equation is Abelian. The cubics for the roots in the other sets are obtained by substituting y_2, y_3, y_4 for y_1, and as we see by the above discussion, y_1, y_2, y_3, y_4 are the roots of a quartic whose coefficients are rational functions of the coefficients of $f(x)$. The same method clearly applies when p and m are equal to any other integers whatsoever.

243. **Solution of a Particular Abelian Equation.**—When $m = 1$, so that the n roots of an Abelian equation $f(x) = 0$ consist of one set only, the equation can be solved by radicals.

In this case all the roots may be expressed in terms of any one root x_1, as follows :—

$$x_1,\ \theta(x_1),\ \theta^2(x_1), \ldots \theta^{n-1}(x_1), \text{ with } \theta^n(x_1) = x_1.$$

Take $\psi_r(x_1) = \{x_1 + \alpha^r\theta(x_1) + \alpha^{2r}\theta^2(x_1) + \ldots + \alpha^{(n-1)r}\theta^{n-1}(x_1)\}^n$, where α is a special or primitive root of $x^n - 1 = 0$, so that the other roots are $\alpha^0, \alpha^2, \alpha^3, \ldots \alpha^{n-1}$, and if m is less than n

$$\alpha^0 + \alpha^m + \alpha^{2m} + \ldots \alpha^{(n-1)m} = 0.$$

Substituting any other root $x_{p+1} = \theta^p(x_1)$ for x_1,

$$\begin{aligned}\psi_r(x_{p+1}) &= \{\theta^p(x_1) + \alpha^r\theta^{p+1}(x_1) + \alpha^{2r}\theta^{p+2}(x_1) + \ldots \\ &\qquad + \alpha^{(n-p)r}\theta^n(x_1) + \ldots + \alpha^{(n-1)r}\theta^{p+n-1}(x_1)\}^n \\ &= \{\alpha^{r(n-p)}(x_1) + \alpha^r\theta(x_1) + \alpha^{2r}\theta^2(x_1) + \ldots \alpha^{(n-1)r}\theta^{n-1}(x_1)\}^n \\ &= \psi_r(x_1),\end{aligned}$$

$$\therefore \psi_r(x_1) = \psi_r(x_2) = \ldots = \psi_r(x_n) = \frac{1}{n}\sum_{s=1}^{n}\psi_r(x_s) = \therefore \text{ a rational}$$

integral function u_r of the coefficients of $f(x)$ and α^r.

Taking n^{th} roots, we derive n equations by giving r the values $0, 1, 2, \ldots n-1$, all of the form

$$x_1 + \alpha^r\theta(x_1) + \alpha^{2r}\theta^2(x_1) + \ldots + \alpha^{(n-1)r}\theta^{n-1}(x_1) = \sqrt[n]{u_r},$$

where $\sqrt[n]{u_r}$ is some one of the n^{th} roots of u_r.

For $r = 0$ the left side becomes $x_1 + x_2 + x_3 + \ldots + x_n = -$ (coefficient of x^{n-1} in $f(x)$) / (coefficient of x^n) $= A_0$. If we add both sides of these n equations, remembering that

$$\alpha^0 + \alpha^m + \alpha^{2m} + \ldots + \alpha^{(n-1)m} = 0,$$

we obtain $nx_1 = A_0 + \sqrt[n]{u_1} + \sqrt[n]{u_2} + \ldots + \sqrt[n]{u_{n-1}}$.

If we multiply the equations by $\alpha^0, \alpha^{-m}, \alpha^{-2m}, \ldots \alpha^{-(n-1)m}$ respectively and add we obtain

$$nx_{m+1} = m\theta^m(x_1) = A_0 + \alpha^{-m}\sqrt[n]{u_1} + \alpha^{-2m}\sqrt[n]{u_2} + \ldots + \alpha^{-(n-1)m}\sqrt[n]{u_{n-1}}.$$

The value of the radical $\sqrt[n]{u_r}$ which has to be taken with $\sqrt[n]{u_1}$ in these equations is given by $\sqrt[n]{u_r} = A_r\ (\sqrt[n]{u_1})^r$, where A_r is a rational function of the coefficients of $f(x)$ and α, for

$$\frac{\sqrt[n]{u_r}}{(\sqrt[n]{u_1})^r} = \frac{x_1 + \alpha^r\theta(x_1) + \alpha^{2r}\theta^2(x_1) + \ldots + \alpha^{(n-1)r}\theta^{n-1}(x_1)}{\{x_1 + \alpha\theta(x_1) + \alpha^2\theta^2(x_1) + \ldots \alpha^{n-1}\theta^{n-1}(x_1)\}^r} = \chi_r(x_1).$$

Now we saw above that if x_{p+1} is substituted for x_1 in

$$x_1 + \alpha^r\theta(x_1) + \alpha^{2r}\theta^2(x_1) + \ldots \alpha^{(n-1)r}\theta^{n-1}(x_1) \equiv \omega_r(x_1),$$

the result is $\alpha^{(n-p)r}\omega_r(x_1)$.

$$\therefore \chi_r(x_{p+1}) = \frac{\alpha^{(n-p)r}\omega_r(x_1)}{\alpha^{(n-p)r}\{\omega_1(x_1)\}^r} = \chi_r(x_1)$$

$$\therefore \chi_r(x_1) = \chi_r(x_2) = \chi_r(x_3) = \ldots = \chi_r(x_n) = \frac{1}{n}\sum_{s=1}^{n}\chi_r(x_s) = A_r =$$

a rational function of the coefficients of $f(x)$ and α. Hence the general expression for a root x_{m+1} may be written

$$nx_{m+1} = A_0 + y + A_2y^2 + \ldots + A_{n-1}y^{n-1},$$

where $y = \alpha^{-m}\sqrt[n]{u_1}$ which has n values only. Hence Abelian equations of the above type are solvable by radicals.

244. **Second method of solving an Abelian Equation $f(x) = 0$ when all the roots form one group, and when also the degree n of $f(x) = mp$.**—For shortness we take the

case when $p = 3$, $m = 4$, and the general method of proof follows obviously.

Arrange the 12 roots in 4 sets as follows :—

$$\begin{aligned} y_1 &= x_1 + x_5 + x_9 = x_1 + \theta^4(x_1) + \theta^8(x_1), \\ y_2 &= x_2 + x_6 + x_{10} = \theta(x_1) + \theta^5(x_1) + \theta^9(x_1), \\ y_3 &= x_3 + x_7 + x_{11} = \theta^2(x_1) + \theta^6(x_1) + \theta^{10}(x_1), \\ y_4 &= x_4 + x_8 + x_{12} = \theta^3(x_1) + \theta^7(x_1) + \theta^{11}(x_1). \end{aligned}$$

As in Art. 242, taking θ^4 for θ, x_1, $\theta^4(x_1)$, $\theta^8(x_1)$ are roots of a cubic equation whose coefficients are rational functions of y_1, and y_1, y_2, y_3, y_4 are roots of a quartic whose coefficients are rational functions of the coefficients of $f(x)$. But in this case the quartic is also Abelian of the same type, for we shall prove $y_2 = \phi(y_1)$, $y_3 = \phi(y_2)$, $y_4 = \phi(y_3)$, $y_1 = \phi(y_4)$, where ϕ is a rational function of the coefficients of $f(x)$. If r is any integer we have

$$\begin{aligned} y_2 y_1^r &= (x_2 + x_6 + x_{10})(x_1 + x_5 + x_9)^r = \{\theta(x_1) + \theta^5(x_1) + \theta^9(x_1)\} \\ &\qquad\qquad \{x_1 + \theta^4(x_1) + \theta^8(x_1)\}^r \\ &= \chi(x_1) \\ &= (x_6 + x_{10} + x_2)(x_5 + x_9 + x_1)^r = \{\theta(x_5) + \theta^5(x_5) + \theta^9(x_5)\} \\ &\qquad\qquad \{x_5 + \theta^4(x_5) + \theta^8(x_5)\}^r \\ &= \chi(x_5). \end{aligned}$$

Similarly $y_2 y_1^r = \chi(x_9)$.

In the same way $y_3 y_2^r = \chi(x_2) = \chi(x_6) = \chi(x_{10})$,

$y_4 y_3^r = \chi(x_3) = \chi(x_7) = \chi(x_{11})$, $y_1 y_4^r = \chi(x_4) = \chi(x_8) = \chi(x_{12})$,

as in the latter case $x_1 = \theta(x_{12})$, $x_5 = \theta(x_4)$, $x_9 = \theta(x_8)$.

Hence

$$y_2 y_1^r + y_3 y_2^r + y_4 y_3^r + y_1 y_4^r = \tfrac{1}{3}\{\chi(x_1) + \chi(x_2) + \ldots \chi(x_{12})\} = T_r,$$

where T_r is a rational function of the coefficients of $f(x)$.

Taking $r = 0, 1, 2, 3$, we have

$$\begin{aligned} y_2 + y_3 + y_4 + y_1 &= T_0, \\ y_2 y_1 + y_3 y_2 + y_4 y_3 + y_1 y_4 &= T_1, \\ y_2 y_1^2 + y_3 y_2^2 + y_4 y_3^2 + y_1 y_4^2 &= T_2, \\ y_2 y_1^3 + y_3 y_2^3 + y_4 y_3^3 + y_1 y_4^3 &= T_3, \end{aligned}$$

and hence as in Art. 226, $y_2 = \phi(y_1)$, $y_3 = \phi(y_2)$, $y_4 = \phi(y_3)$, $y_1 = \phi(y_4)$, where ϕ is a rational integral function whose coefficients are rational functions of the coefficients of $f(x)$. Hence the quartic for y_1, y_2, y_3, y_4 is an Abelian equation of the same type as $f(x)$, and similarly the equation of degree m whose roots are the m sums of the roots in each set of p is Abelian of the same type as $f(x)$.

Generally by this method the solution of an Abelian equation of this type may, if n can be broken up into factors, be made to depend on the solution of Abelian equations, all of the same type and lower degree. Thus if $n = 24$, the solution may be made to depend on the solution of an Abelian equation of degree 12 whose coefficients depend on the solution of an Abelian quadratic. The solution of the Abelian equation of degree 12 may be made to depend on the solution of an Abelian sextic whose coefficients depend on the solution of an Abelian quadratic, and lastly the solution of the Abelian sextic on that of an Abelian cubic whose coefficients depend on the solution of an Abelian quadratic.

245. Solution of a Binomial Equation $x^p - 1 = 0$ when p is a prime number.

If α_1 is any root other than unity of $x^p - 1 = 0$, all the roots are included in the series $1, \alpha_1, \alpha_1^2, \ldots \alpha_1^{p-1}$ (Vol. I., Art. 49). Now the roots of $(x^p - 1)/(x - 1) = 0$ are $\alpha_1, \alpha_1^2, \ldots \alpha_1^{p-1}$. Hence we shall prove that we can obtain an integer a such that the remainders when $a, a^2, a^3, \ldots a^{p-1}$ are divided by p are $1, 2, \ldots p-1$ in some order, the remainder however of a^{p-1} being unity. Hence the roots may be written in the form $\alpha_1^a, \alpha_1^{a^2}, \alpha_1^{a^3}, \ldots \alpha_1^{a^{p-1}}$; and so taking $\theta(x) \equiv x^a$, and putting x_1 for α_1^a, they may be written in the form $x_1, \theta(x_1), \theta^2(x_1), \ldots \theta^{p-2}(x_1)$, with

$$\theta^{p-1}(x_1) = x_1^{a^{p-1}} = x_1.$$

Hence the equation $(x^p - 1)/(x - 1) = 0$ is Abelian of the type in which all the roots form one group.

In the following proof of the above statement, all letters denote integers; p is taken to be a prime number; all functions of x are rational and integral with coefficients which are integers, the coefficient of the highest power of x being unity; and the symbol $\equiv$ in $f(x) \equiv \phi(x)$ denotes that the remainders are the same when $f(x)$ and $\phi(x)$ are divided by p, and so in particular $f(x) \equiv 0$ denotes that $f(x)$ is divisible exactly by p, x being of course an integer. Such quasi-equations are called congruencies.

(*a*) If a less than p satisfies $f(x) \equiv 0$, x is called a root of $f(x) \equiv 0$. Any integer $a + mp$ also satisfies $f(x) \equiv 0$, since $a^n \equiv (a + mp)^n$ and $\therefore f(a) \equiv f(a + mp)$; but the term root of $f(x) \equiv 0$ is restricted to the integer which is less than p.

Now $f(x) \equiv 0$ has not more roots than its degree n. For if a_1 is a root, $f(x) = (x - a_1) f_1(x) + R_1$, and as $f(a_1) \equiv 0$, $R_1 \equiv 0$, $\therefore f(x) \equiv (x - a_1) f_1(x)$. If $f(x)$ has a second root a_2, we must have $f_1(a_2) \equiv 0$, and $\therefore$ as before $f_1(x) \equiv (x - a_2) f_2(x)$. Proceeding in this way, if $f(x) \equiv 0$ of degree n has n roots $a_1, a_2, \ldots a_n$, we obtain $f(x) \equiv (x - a_1)(x - a_2) \ldots (x - a_n)$, and so $f(x) \equiv 0$ has no other root, as no other value of x less than p can satisfy $(x - a_1)(x - a_2) \ldots (x - a_n) \equiv 0$.

(*b*) If $f(x)$ of degree n can be broken up into factors $f_1(x)$, $f_2(x)$ of degrees l, and $n - l$ respectively, and if $f(x_1) \equiv 0$ has n root, as each root must satisfy $f_1(x) \equiv 0$ or $f_2(x) \equiv 0$, $f_1(x) \equiv 0$ must have exactly l roots and $f_2(x) \equiv 0$ must have $n - l$.

(*c*) If $1, 2, \ldots p - 1$ be multiplied by any number a less than p, the remainders when $a, 2a, 3a, \ldots (p-1)a$ are divided by p are all different, and must be 1, 2, 3, $\ldots p - 1$ in some order, for if not $la - ma = (l - m)a$ would be divisible by p, which is impossible as l and m are each less than p, and p of course is a prime number. Hence the product of $a, 2a, 3a, \ldots (p-1)a$ $-$ the product of $1, 2, \ldots p - 1$ is divisible by p,

$$\therefore 1, 2, 3 \ldots (p-1)(a^{p-1} - 1) \equiv 0, \therefore a^{p-1} - 1 \equiv 0,$$

and therefore the roots of $x^{p-1} - 1 \equiv 0$ are $1, 2, 3, \ldots p - 1$.

(*d*) If we take any number a less than p and therefore of course prime to it as p is a prime number, some one of the remainders must recur when $a, a^2, a^3, \ldots$ are divided by p as there are only $p-1$ possible remainders, and so $a^{n+\nu} \equiv a^\nu$, $\therefore a^\nu(a^n-1) \equiv 0$, $\therefore a^n - 1 \equiv 0$, $\therefore a^{n+r} - a^r \equiv 0$, $\therefore$ the first $n-1$ remainders recur in the same order perpetually. Now by (c) $a^{p-1} - 1 \equiv 0$, $\therefore$ $n = p - 1$ or is less than it. If it is less it must be a divisor of $p - 1$. For if $p - 1 = mn + r$ where r is less than n, as $a^{p-1} \equiv 1$, $\therefore a^{mn}a^r \equiv 1$, but $a^{mn} \equiv 1$, as since $a^n \equiv 1$ $\therefore a^{mn} \equiv 1$, and hence if $a^r \equiv l$, as $a^{mn}a^r \equiv 1$, $\therefore l = 1$, $\therefore n$ is not the least integer for which $a^n \equiv 1$.

When a is such that it is a root of $x^n - 1 \equiv 0$, and not a root of any congruency $x^m - 1 \equiv 0$ where m is less than n, it is called a primitive root of $x^n - 1 \equiv 0$, and is such that the remainders when divided by p of $a, a^2, a^3, \ldots a^{n-1}$ are all different and different from unity. The theorem then which we have to establish is that $x^{p-1} - 1 \equiv 0$ has primitive roots.

(e) Express $p - 1$ by its prime factors, say $p - 1 = q^l r^m s^n$, where q, r, s are prime numbers. As $x^{q^l} - 1$ is a factor of $x^{p-1} - 1$, by (b) it has q^l roots, and if any of its roots also satisfy $x^k - 1 \equiv 0$, where k is less than q^l, k by the same reasoning as in (d) must be a factor of q^l; and also as $x^{kl} - 1 \equiv 0$ if $x^k - 1 \equiv 0$ all such roots must satisfy $x^{q^{l-1}} - 1 \equiv 0$. As q^{l-1} is a factor of $p - 1$, $x^{q^{l-1}} - 1 \equiv 0$ has q^{l-1} roots, and so there are q^{l-1} roots of $x^{q^l} - 1 \equiv 0$ and no more which are also roots of binomial congruencies of lower degree, and $\therefore$ there are $q^l - q^{l-1}$ primitive roots of $x^{q^l} - 1 \equiv 0$.

(f) If a is a primitive root of $x^m - 1 \equiv 0$ and b one of $x^n - 1 \equiv 0$, then ab is a primitive root of $x^{mn} - 1 \equiv 0$, if m, n are prime to each other.

Let s be the smallest integer, such that $(ab)^s - 1 \equiv 0$, $\therefore$ $a^s b^s - 1 \equiv 0$, $\therefore a^{ms}b^{ms} \equiv 1$, but $a^{ms} \equiv 1$, $\therefore b^{ms} \equiv 1$, $\therefore ms$ must be a multiple of n, $\therefore s$ is a multiple of n. Similarly s is a multiple of m, $\therefore s$ is a multiple of mn. But $(ab)^{mn} \equiv 1$, $\therefore$ mn is a multiple of s and $\therefore s = mn$.

If now a is a primitive root of $x^{q^l} - 1 \equiv 0$ which we proved must exist in (e), and b one of $x^{r^m} - 1 \equiv 0$, it follows that ab is a primitive root of $x^{q^l r^m} - 1 \equiv 0$. If, further, c is a primitive root of $x^{s^n} - 1 \equiv 0$, it follows that abc is a primitive root of $x^{p-1} - 1 \equiv 0$ when $p - 1 = q^l r^m s^n$. Proceeding in this way, we prove that $x^{p-1} - 1 \equiv 0$ always has primitive roots, and so we establish the theorem that a number a can be found such that the remainders when $a, a^2, a^3, \ldots a^{p-1}$ are divided by p are all different and that the last remainder is unity.

An important example of the theorem at the beginning of this Article is when $p = 17$. The solution, therefore, of the binomial equation $x^{17} - 1 = 0$, or of the problem to inscribe a polygon of 17 equal sides in a circle depends on an Abelian equation of degree 16, all of whose roots form a group, and hence as $16 = 2^4$, it depends by Art. 244 on the solution of quadratic equations only, and so on extraction of square roots only. Hence the geometrical problem may be solved by drawing lines and circles. This equation is given in Vol. I., p. 102, and the arrangement made there of the roots arises by using the integer 3, which is a primitive root of $x^{16} - 1 \equiv 0$ for $p = 17$, and grouping the roots as in Art. 244, taking $\theta(x) \equiv x^3$.

246. **If one root of an irreducible equation is a rational function of one other root, the equation is Abelian.**—If $f(x) = 0$ of degree n is an irreducible equation, and if one root x_2 is a rational function θ of another root x_2 so that $x_2 = \theta(x_1)$, all the roots are so connected, and the equation is Abelian. To prove this, we transform $f(x) = 0$ by the substitution $y = \theta(x)$ to an equation of the same degree $\phi(y) = 0$. As

$\phi(y) = 0$ has a root $= x_2$, it must have all its roots the same as those of $f(x) = 0$, and in fact be equivalent to $f(x) = 0$, for otherwise by finding the least common factor of $f(x)$ and $\phi(x)$, $f(x)$ would be reducible. Hence all the roots of $\phi(y) = 0$ are also roots of $f(x) = 0$, and hence every root x_α of $f(x)$ is connected uniquely with some other root x_β by the equation $x_\beta = \theta(x_\alpha)$. Starting then with x_1, we have $x_2 = \theta(x_1)$, then take

$$x_3 = \theta(x_2) = \theta^2(x_1),$$

and so on until we get $x_1 = \theta^p(x_1)$, and thus obtain a cycle of p roots. Now as $x_1 = \theta^p(x_1)$, the equation $x = \theta^p(x)$ is either an identity or has a root x_1 common with $f(x) = 0$, and therefore as $f(x)$ is irreducible, as above it must be equivalent to $f(x) = 0$. Thus in either case $x = \theta^p(x)$ is satisfied also by $x_2, x_3, \ldots, x_n$. If we start, therefore, with a root x_{p+1} not in the cycle obtained, and find $x_{p+2} = \theta(x_{p+1})$, $x_{p+3} = \theta(x_{p+2})$, and so on, this new cycle must end with $x_{2p} = \theta^{p-1}(x_{p+1})$, for $x_{p+1} = \theta^p(x_{p+1})$. Were it to involve q roots only, and so involve the equation $x_{p+1} = \theta^q(x_{p+1})$ with q less than p, then as before all the roots would satisfy $x = \theta^q(x)$, and the first cycle should have ended with $x_q = \theta^{q-1}(x_1)$, but it has been assumed not to do so, and therefore $q = p$ for it cannot be either greater or less than it. Proceeding in this way, we divide all the roots into m cycles of p each, and we must have $n = mp$, and the equation is Abelian. All the cycles must contain different roots, as if two had one common the next in order in each cycle would also be common, and so proceeding in order all the roots in the two cycles would be equal, and so the root with which the second cycle was started would have occurred previously, which was assumed not to be the case. Also, of course, no two roots of $f(x) = 0$ can be equal, as it is irreducible.

MISCELLANEOUS NOTES.

Page 4. The number of inversions of order consist of the number of suffixes in front of the suffix 1, the number in front of 2 and greater than it, the number in front of 3 and greater than it, and so on. Thus the number of inversions is equal also to the number of consecutive transpositions required to bring first the suffix 1 into the leading position, then 2, then 3, and so on.

Page 17. Laplace's development of a determinant. We represent a determinant by $(a_\alpha, b_\beta, c_\gamma, \ldots)$, implying that the first row is $a_\alpha, b_\alpha, c_\alpha, \ldots$, the second $a_\beta, b_\beta, c_\beta, \ldots$, and so on, so that $a_\alpha b_\beta c_\gamma \ldots$ is the diagonal term, and $\alpha, \beta, \gamma, \ldots$ the standard order of suffixes, which need not be 1, 2, 3, ... We shall prove Laplace's development of a determinant in terms of the minors formed from any number of rows or columns by showing that $\Delta = (a_1 b_2 c_3 d_4 e_5 f_6 g_7)$ can be expanded in terms of the minors formed from the first three columns, and the method used will obviously apply generally. Take any combination of the 7 suffixes taken three at a time, attach any such triad in order to a, b, c and the rest in order to d, e, f, g, thus getting, say, $a_2 b_5 c_6 . d_1 e_3 f_4 g_7$. The number of inversions in this term is due to suffixes attached to a, b, c being greater than suffixes attached to d, e, f, g, and in this case is 7. Now permute the suffixes 2, 5, 6 attached to a, b, c, keeping the rest fixed. The additional inversions so introduced into any term, say $a_5 b_6 c_2 . d_1 e_3 f_4 g_7$, are also the inversions in $a_5 b_6 c_2$ considered as a term of (a_2, b_5, c_6), in this case 2. Attach the sign $(-1)^2$ to $a_5 b_6 c_2$, and we have a term of (a_2, b_5, c_6). Doing the same to every term arising from a permutation of 2, 5, 6 and adding, we get $(-1)^7 . (a_2, b_5, c_6) d_1 e_3 f_4 g_7$. Now permute in every possible way the suffixes 1, 3, 4, 7, leaving (a_2, b_5, c_6) unaltered, and in a similar way the terms of Δ so arising will give us

$(-1)^7(a_2, b_5, c_6)(d_1, e_3, f_4, g_7)$. Every combination of the 7 suffixes taken three at a time will give rise to a similar product of two minors, and the sign to be attached to the product is $(-1)^m$ where m is the number of inversions owing to the suffixes attached to a, b, c being greater than those attached to d, e, f, g.

Page 28. That a determinant Δ written as in Art. 141 is the product of two determinants may be seen as follows: We note that αs are multiplied by as, βs by bs, and so on, and we shall call a vertical column formed by terms such as $b_1\beta_2, b_2\beta_2, b_3\beta_2$ a vertical column of similar terms. When we take a column of similar vertical terms from the first column of Δ, we must take with it a dissimilar column of similar terms from the next column of Δ, and then a column of similar terms from the third column of Δ which is dissimilar to the previous two, and so on generally. We note in the determinant so formed which contains (a_1, b_2, c_3) as a factor multiplied by a term of $(\alpha_1, \beta_2, \gamma_3)$, that every inversion in order of columns considered as columns of (a_1, b_2, c_3) is accompanied by a similar inversion in the suffixes of the term of $(\alpha_1, \beta_2, \gamma_3)$, so that the number of consecutive transpositions of columns to obtain (a_1, b_2, c_3) is precisely that required to give the proper sign to the term of $(\alpha_1, \beta_2, \gamma_3)$. Every term of $(\alpha_1, \beta_2, \gamma_3)$ with proper sign arises in this way multiplied by (a_1, b_2, c_3), and we see that $\Delta = (a_1, b_2, c_3)(\alpha_1, \beta_2, \gamma_3)$, and that a similar method of proof applies generally.

Page 87. When $U = 0$ and $V = 0$ have two roots α, β common, as $\alpha \frac{\partial R}{\partial a_p} = \frac{\partial R}{\partial a_{p+1}}$, and $\beta \frac{\partial R}{\partial a_p} = \frac{\partial R}{\partial a_{p+1}}$, therefore $(\alpha - \beta)\frac{\partial R}{\partial a_p} = 0$, therefore $\frac{\partial R}{\partial a_p} = 0$, and therefore $\frac{\partial R}{\partial a_{p+1}} = 0$, as $\alpha \frac{\partial R}{\partial a_p} = \frac{\partial R}{\partial a_{p+1}}$.

NOTE A.

DETERMINANTS.

The expressions which form the subject-matter of Chapter XIII. were first called "determinants" by Cauchy, this name being adopted by him from the writings of Gauss, who had applied it to certain special classes of these functions, viz. the discriminants of binary and ternary quadratic forms. Although Leibnitz had observed in 1693 the peculiarity of the expressions which arise from the solution of linear equations, no further advance in the subject took place until Cramer, in 1750, was led to the study of such functions in connexion with the analysis of curves. To Cramer is due the rule of signs of Art. 128. During the latter part of the eighteenth century the subject was further enlarged by the labours of Bezout, Laplace, Vandermonde, and Lagrange. In the nineteenth century the earliest cultivators of this branch of mathematics were Gauss and Cauchy; the former of whom, in addition to his investigations relative to the discriminants of quadratic forms, proved, for the particular cases of the second and third order, that the product of two determinants is itself a determinant. To Cauchy we are indebted for the first formal treatise on the subject. In his memoir on *Alternate Functions*, published in the *Journal de l'École Polytechnique*, vol. x, he discusses determinants as a particular class of such functions, and proves several important general theorems relating to them. A great impulse was given to the study of these expressions by the writings of Jacobi in Crelle's *Journal*, and by his memoirs published in 1841. Among many mathematicians who have advanced this subject in more recent years may be mentioned Brioschi, Hermite, Hesse, Joachimsthal, Cayley, Sylvester, and Salmon. There is now no department of mathematics, pure or applied, in which the employment of this calculus is not of great assistance, not only furnishing brevity and elegance in the demonstration of known properties, but even leading to new discoveries in mathematical science. Among recent works which have rendered the subject accessible to students may be mentioned Spottiswoode's *Elementary Theorems relating to Determinants*, London, 1851; Brioschi's *La teorica dei Determinanti*, Pavia, 1854; Baltzer's *Theorie und Anwendung der Determinanten*, Leipzig, 1864; Dostor's *Éléments de la théorie des Déterminants*, Paris, 1877; Scott's *Theory of Determinants*, Cambridge, 1880; and the chapters in Salmon's *Lessons Introductory to the Modern Higher Algebra*, Dublin, 1876. For further information on the history of this subject the reader is referred to Muir's *Theory of Determinants in the historical order of its development*, London, 1890. In Salmon's *Higher Algebra* there are short historical notes on Eliminants, Invariants, Covariants, and Linear Transformations, as well as on Determinants.

NOTE B.

COMBINED FORMS.

We give here, as an Appendix to Chapter XVIII., an enumeration of the concomitants of two quartics U and V. For this purpose it is convenient to use the notation $(\psi, \phi)^p$ for $(1, 2)^p\,\phi_1\psi_2$, when the distinction between the variables is removed. In this notation we have the sixteen concomitants $(U_x, V_x)^p$, $(U_x, H'_x)^p$, $(V_x, H_x)^p$, $(H_x, H'_x)^p$, when p has the four values 1, 2, 3, 4, viz. twelve covariants and four invariants; but of these Sylvester has reduced (H_x, H'_x) and $(H_x, H'_x)^2$, so that only ten independent covariants are obtained in this way; we have, however, to add the four quadratic covariants $(G_x, V_x)^4$, $(G'_x, U_x)^4$, $(H_x, G'_x)^4$, $(H'_x, G_x)^4$. These are the fourteen special covariants of this system (Gordan, *Math. Ann.* II. 275). To this list are to be added the five forms belonging to each quartic separately, viz. U_x, H_x, G_x, I, J, and V_x, H'_x, G'_x, I', J'. Hence there are in all twenty-eight forms made up as follows:— eight invariants, eight quadric, seven quartic, and five sextic covariants. The theory of two binary quartics may be reduced to that of three ternary quadrics as a particular case. See *Quarterly Journal of Mathematics*, vol. x. p. 239.

The Table which follows gives the number of forms of the combined systems from I., I. to IV., IV.:—

	I.	II.	III.	IV.
I.	3	5	13	20
II.		6	15	18
III.			26	61
IV.				28

NOTE C.

THE QUINTIC AND ITS CONCOMITANTS.

Gordan fixes the number of independent concomitants as twenty-three, which may be derived as follows:—the first fourteen, viz. four invariants, four linear covariants, three quadratic covariants, and three cubic covariants come from the covariants I_x of the second degree and J_x of the third degree considered as a distinct combined system in the manner of Art. 191. One reduction, however, in the number there obtained (viz. fifteen, the number of irreducible forms of the combined system) occurs in this case, for the resultant of I_x and J_x, or $R(I_x J_x)$, is the same as the discriminant of J_x, or $\Delta(J_x)$, both leading to the same invariant of the twelfth order. In addition to the fourteen thus obtained, the remaining nine concomitants are defined as follows, K_x being used to denote the Hessian of J_x:—

Quartic Covariants : $I_D(H_x) \equiv Q_x$, $J(I_x, Q_x)$;
Quintic Covariants : U_x, $J(U_x, I_x)$, $J(U_x, K_x)$;
Sextic Covariants : H_x, $J(I_x, H_x)$;
Septimic Covariant : $J(H_x, J_x)$;
Nonic Covariant : $J(U_x, H_x)$.

The foregoing results are collected in the following Table, where p signifies the degree in the variables, ϖ the order in the coefficients of the quintic, and N the number of concomitants of each degree :—

p	ϖ				N
0	4	8	12	18	4
1	5	7	11	13	4
2	2	6	8		3
3	3	5	9		3
4	4	6			2
5	1	3	7		3
6	2	4			2
7	5				1
9	3				1

Adopting the definitions of the invariants given by Clebsch and Gordan, and implied in the following equation (see Art. 190), the connexion between the four invariants of the quintic is established as follows by Gordan :—

$$-J^2(I_x, K_x) = I_4K_x^2 - 2I_8I_xK_x + I_{12}I_x^2;$$

also $$\tfrac{1}{3}I_D(J_x) = L_x \equiv L_0x + L_1y.$$

Now, substituting L_1 and $-L_0$ for x and y in I_x, K_x, and in $J(I_x, K_x)$ we find $$-I_{18}^2 = F(I_4, I_8, I_{12}),$$

since $$R(I_x, L_x) = 12I_{12} - 16I_4I_8,$$

$$R(K_x, L_x) = I_8^2 - I_4I_{12}.$$

Thus I_{18} is defined, and its square expressed in terms of the other invariants which are not skew.

NOTE D.

THE SEXTIC AND ITS CONCOMITANTS.

THE first sixteen of the twenty-six forms of the sextic come from I_x and L_x treated as a combined system (Art. 217). In this way we obtain all the invariants, quadratic covariants, and quartic covariants. There are in general eighteen forms in the combination of a quartic and quadratic; but, in this special case, owing to the nature of the coefficients, the invariant D_1, which is an invariant I_8 of the sextic, is expressible in terms of the invariants I_2, I_4, I_6, in the form $I_8 = pI_4^2 + qI_2I_6$: also the covariant sextic of I_x is reducible to those which occur in the enumeration which follows. It should be noticed that all these forms are even in the variables, $n\varpi - 2\kappa$ being even for the sextic.

The following is a complete enumeration of the covariants :—

Quadrics : $L_x \equiv I_D(U)$, $M_x \equiv L_D(I_x)$, $N_x \equiv M_D(I_x)$,
$J(L_x, M_x)$, $J(L_x, N_x)$, $J(M_x, N_x)$.

Quartics : I_x, $H(I_x)$, $J(I_x, L_x)$, $J(I_x, M_x)$, $J(I_x, N_x)$.

Sextics : U, J_x, $J(U, L_x)$, $J(U, M_x)$, $J(J_x, L_x)$.

Octavics : H_x, $J(U, I_x)$, $J(H_x, L_x)$.

Decimic : $J(I_x, H_x)$.

Duodecimic : G_x.

These results are collected in the following Table, in which p is the degree of the concomitant, ϖ the order in the coefficients, and N the number of each kind :—

p	ϖ						N
0	2	4	6	10	15		5
2	3	5	7	8	10	12	6
4	2	4	5	7	9		5
6	1	3	4	6	6		5
8	2	3	5				3
10	4						1
12	3						1

The skew invariant Ω_{15} (Art. 217) of the combined system I_x and L_x, being the skew invariant I_{15} of the sextic, its square can similarly be expressed in terms of the invariants of an even degree of the sextic.

It will be noticed that there are two covariants of the sixth degree in the variables, and of the sixth order in the coefficients ; this is the first instance in which there are two irreducible seminvariants of the same order and weight in the binary system.

It may be observed that if the ternary form of any three of the quadratic covariants be taken as lines of reference, the sextic will be represented by a cubic and conic combined, such that every coefficient in the equation of either curve is an invariant of the sextic.

NOTE E.

In illustration of the principles of Art. 208, and in order to account for invariants which being linearly independent are not algebraically so, we add here some examples taken from various parts of this volume.

(1). In the case of two cubics (Art. 192), we have nine equations, including $\lambda\mu' - \lambda'\mu = M$, to eliminate $\lambda, \mu, \lambda', \mu'$: hence five invariants, which form a list algebraically complete, since the combinants P and Q of Art. 192 can, when squared, be expressed in terms of the five which are of an even weight (cf. Ex. 29, p. 213). It is seen therefore that, although there are seven linearly independent invariants, only five are algebraically independent.

(2). In the case of the quintic (see Note C) there are seven equations to eliminate $\lambda, \mu, \lambda', \mu'$, giving three algebraically independent invariants. Of the four linearly independent invariants one is, when squared, expressible in terms of the others, for

$$I_{18}^2 = F(I_4, I_8, I_{12}).$$

(3). In the case of the sextic there are eight equations to eliminate $\lambda, \mu, \lambda', \mu'$, giving four invariants algebraically independent. The sextic has, however, five linearly independent invariants, these being connected by a relation of the form

$$I_{15}^2 = F(I_2, I_4, I_6, I_{10}),$$

where I_{15} is the skew invariant of the combined system of the quadratic I_x and the quartic L_x (Note D), whose square has been expressed in terms of the other invariants of an even weight (Art. 217).

It is instructive to consider the absolute invariants of a binary quantic from a geometrical point of view. If the n roots be

$$\alpha, \beta, \gamma, \rho_1, \rho_2, \ldots \rho_{n-3},$$

there are $n-3$ independent anharmonic ratios which may be represented as follows:—

$$(\alpha, \beta, \gamma, \rho_1), \quad (\alpha, \beta, \gamma, \rho_2), \ldots (\alpha, \beta, \gamma, \rho_{n-3}).$$

All the anharmonic ratios can be rationally expressed in terms of these, and, since they are unaltered by any linear transformation (Art. 38, Vol. I.), they are $n-3$ independent, irrational, absolute invariants. These results must, moreover, be implied by the $n+1$ equations connecting the old and new coefficients $a_0, a_1, \ldots a_n$ and $A_0, A_1, \ldots A_n$, since they embrace all the general consequences of every linear transformation of the quantic, however expressed.

INDEX TO VOLUME II.

[The figures appended refer to the pages.]

END OF VOL. II.

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